# Probability Prelim Exam

January 2017

**Instructions (Read before you begin)**

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

**Exam Problems:**

1. Consider an array $\{X_{i,j}\}$ of random variables such that, for every $n \geqslant 1$, the collection $X_{1,n}, \ldots, X_{n,n}$ is i.i.d. with common distribution

$$\mathrm{P}\{X_{1,n} = 1\} = \mathrm{P}\{X_{1,n} = -1\} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right);$$

   and $\mathrm{P}\{X_{1,n} = n^2\} = n^{-2}$. Let $S_n := X_{1,n} + \cdots + X_{n,n}$. Prove that $S_n/\sqrt{n} \Rightarrow \mathrm{N}(0\,, \sigma^2)$ for some $\sigma \in (0\,, \infty)$, although $\mathrm{E}(S_n/\sqrt{n}) \to \infty$ as $n \to \infty$.

2. Consider a one-dimensional simple symmetric random walk: $S_n = x + \sum_{k=1}^{n} X_k$, where $x$ is where the walk starts and $\{X_n : n \geqslant 1\}$ are i.i.d. with

$$\mathrm{P}(X_n = 1) = \mathrm{P}(X_n = -1) = 1/2.$$

   (a) Compute the probability that starting at $x = 0$ the random walk will hit $-1$ before it hits $b \in \mathbb{N}$. Deduce from this that the random walk is recurrent, i.e. starting at 0 the random walk will return to 0 with probability 1.

   (b) Compute the mean time it takes the random walk to go from $x = 0$ to $-1$. Deduce from this that the random walk is null recurrent, i.e. starting at 0, the mean time it takes to return to 0 is infinite.

   (Hint: Prove first that when $x = 0$ both $S_n$ and $S_n^2 - n$ are martingales.)

3. Let $X_1, X_2, \ldots$ be a collection of i.i.d. random variables, and define $S_n := X_1 + \cdots + X_n$ for all $n \geqslant 1$. Suppose

$$\mathrm{P}\{X_1 = k\} = \frac{3}{\pi^2 k^2} \qquad \text{for all } k = \pm 1, \pm 2, \ldots.$$

Prove that $S_n/n$ converges weakly to a non-constant random variable as $n \to \infty$.

4. Suppose $X, Y \in L^1(P)$ satisfy $E[X|Y] = Y$ and $E[Y|X] = X$ almost surely. Prove that $X = Y$ almost surely. (Hint: Prove first that $E[X - Y; X < q < Y] = 0$ for all $q$.)

5. Let $p_n$ and $q_n$ denote two probability mass functions on $\mathbb{Z}^n$, for every integer $n \geqslant 1$. That is, for all $n \geqslant 1$:

   (a) $p_n(\boldsymbol{x}) \geqslant 0$ and $q_n(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \mathbb{Z}^n$; and

   (b) $\sum_{\boldsymbol{x} \in \mathbb{Z}^n} p_n(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in \mathbb{Z}^n} q_n(\boldsymbol{x}) = 1$.

   Now suppose $\{X_i\}_{i=1}^{\infty}$ is a collection of random variables such that $(X_1, \ldots, X_n)$ has mass function $p_n$ for every $n \geqslant 1$. That is,

$$\mathrm{P}\{X_1 = x_1, \ldots, X_n = x_n\} = p_n(x_1, \ldots, x_n) \qquad \text{for all } (x_1, \ldots, x_n) \in \mathbb{Z}^n.$$

   (a) Prove that $p_n(X_1, \ldots, X_n) > 0$ a.s. for all $n \geqslant 1$.

   (b) Prove that $\{Z_n\}_{n=1}^{\infty}$ is a martingale, where, for all $n \geqslant 1$,

$$Z_n := \frac{q_n(X_1, \ldots, X_n)}{p_n(X_1, \ldots, X_n)},$$

   almost surely on the P-measure-one event $\{p_n(X_1, \ldots, X_n) > 0\}$, and $Z_n := 0$ otherwise.

   (c) Prove that $\lim_{n\to\infty} Z_n$ exists a.s. and is finite a.s. (This is a starting point for likelihood-ratio testing in classical statistics.)

6. If $X_1, X_2, \ldots$ are independent standard normal random variables, then find non-random sequences $a_n, b_n \to \infty$ such that $a_n \min_{1 \leqslant i \leqslant n} X_i + b_n$ converges weakly. Identify the limiting distribution.

7. Suppose $X_1, X_2, \ldots$ are independent mean-zero random variables that satisfy the condition $\sum_{i=1}^{\infty} \mathrm{Var}(X_i) < \infty$.

   (a) Prove that $\sum_{n=1}^{\infty} X_n$ exists a.s. and is finite a.s.

   (b) Use part (a) to prove that if $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. with $\mathrm{P}\{\varepsilon_1 = \pm 1\} = 1/2$, then the random harmonic series $\sum_{i=1}^{\infty} (\varepsilon_i / i)$ converges a.s. and is finite a.s.

   (c) Is $\sum_{i=1}^{\infty} (\varepsilon_i / i)$ in $L^1(\mathrm{P})$?

8. Suppose that $X_1, X_2, \ldots$ are i.i.d. exponential random variables with parameter $\lambda > 0$. Prove that $\overline{\lim}_{n\to\infty} (X_n / \log n) = \lambda^{-1}$ and $\underline{\lim}_{n\to\infty} (\log X_n / \log n) = -1$.

9. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathrm{P}\{X_1 = 1\} = p$ and $\mathrm{P}\{X_1 = -1\} = q$, where $p \notin \{0\,, \frac{1}{2}\,, 1\}$ and $q = 1 - p$. Define $S_n := X_1 + \cdots + X_n$ for all $n \geqslant 1$ and consider the stopping time, $T := \inf\{n \geqslant 1 : |S_n| = 5\}$, where $\inf \varnothing := \infty$. Prove that $T \in L^1(\mathrm{P})$ and compute $\mathrm{E}(T)$. (HINT: Begin by proving that $M_n := (q/p)^{S_n}$ and $N_n := S_n - (p-q)n$ are martingales.)

10. Let $X$ and $Y$ be two real-valued random variables. Let $f$ and $g$ be two non-decreasing measurable functions such that $f(X)$ and $g(Y)$ are integrable. Prove that

$$E[f(X)g(Y)] \geqslant E[f(X)]E[g(Y)].$$