Preliminary Exam, Numerical Analysis, August 2011

Instructions: This exam is closed books, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1(**Hermitian Matrix**).
Let $A \in \mathbb{C}^{m\times m}$ be hermitian:
a) Prove that all eigenvalues of A are real.
b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Problem 2(**Induced Matrix Norm**).
a) Give the definition of an induced matrix norm.
b) Show that the Frobenius norm $||\cdot||_F$ is not induced by any vector norm.

Problem 3(**Trapezoidal Rule**).
a) State the formula for the composite trapezoidal rule $I_n(f)$ to approximate the integral $I(f) = \int_a^b f(x)dx$
b) Bound the error in $I_n(f)$ applied to the following integral:

$$\int_0^{\pi/2} \cos(x)dx$$

(You do not need to derive the error formula.)

Problem 4(**Numerical Quadrature**).
Show that there is no set of nodes $x_1, x_2, ..., x_n$ and coefficients $\alpha_1, \alpha_2, ..., \alpha_n$ such that the quadrature rule

$$\sum_{j=1}^{n} \alpha_j f(x_j)$$

exactly equals to the integral $\int_a^b f(x)w(x)dx$ for all polynomials $f(x)$ of degree less than or equal to $2n$. $w(x)$ is the weight function.

Problem 5(**Existence and Uniqueness of the Interpolating Polynomial**).
State the theorem about the existence and uniqueness of the interpolating polynomial.
Give the proof. (You can consider any proof of your choice.)

Problem 6(**Linear Multistep Methods**).
a) Define the linear multistep method (give formula). Explain what we mean by its region of absolute stability.
b) Show that the region of absolute stability for the trapezoidal method is the set of all complex $h\lambda$ with Real$(\lambda)< 0$.

Problem 7(**CFL Condition**).
a) Explain what we mean by the CFL (Courant-Friedrichs-Lewy) Condition
b) Obtain CFL condition for the upwind scheme (explicit) below:

$$u_{i,j+1} = (1-\lambda)u_{i,j} + \lambda u_{i-1,j},$$

where $i = 1, 2, ..., N+1$ and $j = 0, 1, 2, ..., M-1$

$$\lambda = \frac{a\Delta t}{h}, \quad \Delta t = T/M,$$

and a is a positive constant.The initial condition is $u_{i,0} = g_i$

Problem 8(**Stability of the Scheme**).
Using the von Neumann method investigate the stability of the implicit upwind scheme:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} - \frac{u_{m+1}^{n+1} - u_m^{n+1}}{h} = f_m^n,$$

$$u_m^0 = g_m, \quad m = 0, \pm 1, \pm 2, ..., \quad n = 0, 1, ..., [T/\Delta t] - 1.$$

Comment on the CFL condition for this scheme.