Preliminary Exam, Numerical Analysis, August 2019

Instructions: This exam is closed book, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1 ((**Hermitian Matrix**).
Let $A \in \mathrm{C}^{m\times m}$ be hermitian:
a) Prove that all eigenvalues of A are real.
b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Problem 2. (**Matrix Factorizations**).
Let A be a nonsingular square real matrix and let $A = QR$ and $A^TA = H^TH$ be QR and Cholesky factorizations, respectively, with the usual normalizations r_{jj}, $h_{jj} > 0$. Is it true or false that $R = H$? Justify your answer.

Problem 3. (**Singular Value Decomposition**).
a) Consider $A \in \mathrm{C}^{m\times n}$. Define what we mean by the singular value decomposition of A.
b) Show that the rank of A is r, the number of nonzero singular values.

Problem 4. (**Jacobi Method**).
a) State the Jacobi method for the solution of the linear system $Ax = b$, where $A \in \mathrm{R}^{m\times m}$
b) Show that if A is a strictly diagonally dominant matrix by rows, the Jacobi method is convergent for any $x^{(0)}$.

Problem 5. (**Interpolation**).
a) State the theorem about the existence and uniqueness of interpolating polynomial. Give a proof. (You can consider any proof of your choice).
b) Let $f(x) = x^2 + 5x + 1$. Find the polynomial of degree 3 that interpolates the values of f at $x = -1, 0, 1, 2$.

Problem 6. (**Linear Multistep Methods**).
Give the definition of linear multistep method (give formula).
a) State a necessary and sufficient condition for the linear multistep method to be consistent.
b) State the root condition for the linear multistep method.
c) Construct an example of a consistent but not stable linear multistep method. Justify your answer.

Problem 7. (**Region of Absolute Stability and Example of Linear One-Step Method**).
1. Define a region of absolute stability for the linear multistep method.
2. Find the region of absolute stability for the Backward Euler Method.

Problem 8. (**Upwind Scheme**).
Consider the advection equation

$$u_t - 9u_x = 0, \quad x_L < x < x_R, \quad 0 < t \leq T,$$

where $u(x, 0) = g(x)$, and $u(x_R, t) = u_R(t)$ for $t > 0$
a) Write the Upwind Scheme for this problem
b) What is the local truncation error of the Upwind Scheme?
c) What is the stencil of the scheme? What is the CFL condition for this method?
d) Investigate the stability of the method using Von Neumann Stability Analysis.