DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

January, 2018

Instructions: This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. Each part has 6 questions. To pass the exam, you will have to pass each part. To pass each part, you will have to demonstrate mastery of the material. It will be up to the faculty grading the exam to determine if sufficient understanding of the material is demonstrated. The exam should be viewed as a dialogue, questions asked and questions answered. The exam will not be graded using a numeric system. There is not a set standard of how many questions need to be answered correctly, so please answer as many questions as you can. Not all questions are equally difficult, and the grading will take this into account.

Section A.

1. Let M and N be smooth manifolds. If $f : M \to N$ is continuous, proper, and injective, prove that $f^{-1} : f(M) \to M$ is continuous. Conclude that any proper, injective, immersion $f : M \to N$ is an embedding.

2. Let M and N be smooth manifolds of the same dimension, and suppose M is compact. Prove that if $f : M \to N$ is an immersion then $f^{-1}(y)$ is finite for any $y \in N$.

3. Give a foliation on S^3 whose leaves are each diffeomorphic to S^1. (To describe the foliation, you don't need to give the charts. Just a description of the leaves will do.)

4. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. For a nonzero $v \in \mathfrak{g}$, let L be the line spanned by v. Prove that $\exp|_L : L \to G$ is an immersion.

5. Prove that S^n is orientable.

6. Let M and N be smooth manifolds, and let $f : M \to N$ be smooth. Prove that if θ and ψ are closed differentiable n-forms on N (so that $[\theta]$ and $[\psi]$ are classes in de Rham cohomology) then $[\theta] = [\psi]$ implies $[f^*\theta] = [f^*\psi]$.

Section B.

7. Suppose X and Y are cell complexes and that $A \subseteq X$ is a subcomplex. Let $f : A \to Y$ be continuous and let $Y \sqcup_f X$ be Y with X attached along A via f. Prove that $Y \sqcup_f X$ is homotopy equivalent to a cell complex.

8. Prove $\pi_1(S^n) = 1$ if $n \geq 2$.

9. Let $p : \widetilde{X} \to X$ be a cover of connected spaces. Prove $p_* : \pi_1\widetilde{X} \to \pi_1 X$ is injective.

10. Let Σ_2 be a closed, orientable surface of genus 2. Prove that $\pi_1(\Sigma_2)$ has a normal subgroup G such that $\pi_1(\Sigma_2)/G \cong \mathbb{Z}$.

11. Prove $H_1(\Sigma_2) \cong \mathbb{Z}^4$.

12. Prove $H_c^k(\mathbb{R}^n)$ equals $\mathbb{Z}$ if $k = n$ and 0 if $k \neq n$.