DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions. You need at least 3 completely correct to pass Part A.

1. (a) Define $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by $F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1^2 + x_2^2 - x_3^2 - x_4^2)$. Show that $M = F^{-1}(1, 0)$ is a smooth manifold.
   (b) For each $x = (x_1, x_2, x_3, x_4) \in M$ show that the tangent space T_xM is spanned by $(x_2, -x_1, 0, 0)$ and $(0, 0, x_4, -x_3)$.
   (c) Let $G\colon \mathbb{R}^4 \to \mathbb{R}$ be a smooth map and $g = G|_M$ the restriction of G to M. Show that x is a critical point of g if and only if $\ker F \subset \ker G$.
   (d) If $G(x_1, x_2, x_3, x_4) = x_1 + x_3$ find the critical points of g.

2. (a) Define $f\colon \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x^2, x^3)$. Find f^*dx_1.
   (b) Assume $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth map and for each $(x_1, x_2) \in \mathbb{R}^2$ the derivative is given by the matrix $\begin{pmatrix} 2x_1 + x_2 & x_1 + 3 \\ \frac{x_1}{(x_1^2+x_2^2)^2} & \frac{x_2}{(x_1^2+x_2^2)^2} \end{pmatrix}$. Find g^*dx_1.

3. Let M be a smooth manifold. Show that its tangent bundle TM is orientable.

4. Let $U \subset \mathbb{R}^n$ be a connected, open subset. For all $p, q \in U$ show that there exists a diffeomorphism $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(p) = q$ and ϕ the identity outside of U. (Hint: First assume that U is convex with compact closure and find a vector field with support in U whose flow takes p to q.)

5. (a) Define the DeRham cohomology groups $H^k(M)$ for a smooth manifold M.
   (b) Let N be a compact, smooth, k-dimensional manifold without boundary and $f : N \to M$ a smooth map. Let $\alpha \in \Omega^k(M)$ be a closed k-form on M with $\int_N f^*\alpha \neq 0$. Either show that α represents a non-zero element of $H^k(M)$ or find a counterexample.
   (c) The wedge product defines a map $\Omega^k(M) \times \Omega^l(M)\colon \Omega^{k+l}(M)$ by $(\alpha, \beta) \mapsto \alpha \wedge \beta$. Show that this descends to a map from $H^k(M) \times H^l(M)$ to $H^{k+l}(M)$.

6. Let G be an n-dimensional Lie group. Show that the tangent bundle TG is diffeomorphic to $G \times \mathbb{R}^n$.

B. Answer all of the following questions. These questions are straightforward. If you are familiar with the basic concepts from algebraic topology, then you will be able to answer all of these questions, or almost all of them, correctly. That's the standard by which passing part B will be judged. Do the best that you can.

7. Show that $\pi_1(S^1) = \mathbb{Z}$; that $\pi_1(S^n) = 1$ if $n \geq 2$; that $\pi_1(\Sigma_1 - \{x\}) = F_2$ (where $\Sigma_1 - \{x\}$ is a closed orientable genus 1 surface with a single point removed and F_2 is a free group on two generators); and find $\pi_1(\Sigma_2)$ (where Σ_2 is a closed orientable surface of genus 2).

8. What is the universal cover of $S^1 \vee S^1$?

9. Let X be a path connected space that is locally simply connected. Give the construction of $p : \widetilde{X} \to X$, the universal cover of X. You don't have to define what a universal cover is, or show that your construction is a universal cover. Just state the description of the universal cover for X used in the standard proof that the universal cover for X exists.

10. Let $\{B_n\}_{n\in\mathbb{Z}}$ and $\{C_n\}_{n\in\mathbb{Z}}$ be chain complexes. What is the definition of a *chain map* from $\{B_n\}_{n\in\mathbb{Z}}$ to $\{C_n\}_{n\in\mathbb{Z}}$?

11. Let SX be the suspension of X. Show that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for all n.

12. Prove that if $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open, nonempty, and homeomorphic, then $n = m$.

13. Let A be a retract of X, and let $i : A \to X$ be the inclusion. Prove that $i_* : H_n(A) \to H_n(X)$ is injective for all n.

14. Find all homology and cohomology groups (with integer coefficients) for the following three spaces: $\mathbb{R}P^3$, $\mathbb{C}P^3$, and $S^1 \vee S^3$.

15. Prove that $S^1 \vee S^3$ is not homotopy equivalent to a compact, orientable, 3-manifold without boundary.