UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 4, 2014.

Instructions: This examination has two parts consisting of six problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. Let $f(x)$ be continuously differentiable for all $X \in \mathbf{R}^n$ and let $x_0 \in \mathbf{R}^n \times \mathbf{R}$. Show that there is and $\varepsilon > 0$ and a unique function $y \in \mathcal{C}^1([0,\varepsilon],\mathbf{R}^n)$ that satisfies the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x), \\ x(0) = x_0. \end{cases}$$

A2. Let A be an $n \times n$ real matrix such that $\Re\mathrm{e}\,\lambda < 0$ for all eigenvalues λ of A. Show that there are positive constants c and μ such that any solution $x(t)$ of

$$\begin{cases} \dfrac{dx}{dt} = A\,x, \\ x(0) = x_0. \end{cases}$$

satisfies

$$|x(t)| \le ce^{-\mu t}|x_0| \qquad \text{for all } t \ge 0.$$

A3. Show that for small $|\varepsilon|$, there is a 2π-periodic solution to the forced Duffing's equation

$$\ddot{y} + 3y + \varepsilon y^3 = \cos t.$$

A4. Let A be an $n \times n$ real matrix such that all solutions of $\dot{x} = Ax$ are bounded as $t \to \infty$. Let $B(t)$ be a continuous real $n \times n$ matrix function that satisfies

$$\int_0^\infty |B(s)|\,ds < \infty.$$

Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = Ax + B(t)x \\ \quad x(0) = x_0 \end{cases} \tag{1}$$

(a) Define: the zero solution of (1) is *stable* (also called uniformly *Liapunov Stable*).

(b) Prove that the zero solution of (1) is stable.

[Don't just quote a theorem. Hint: Suppose $g(t)$ and $u(t)$ are nonnegative functions and $c_0 \geq 0$ is a constant that satisfies $u(t) \leq c_0 + \int_0^t g(s)\, u(s)\, ds$ for all $t \geq 0$. Then Gronwall's Inequality implies $u(t) \leq c_0 \exp\left(\int_0^t g(s)\, ds\right)$ for all $t \geq 0$.]

A5. Show that the equation has a non constant periodic solution.

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + 2x = 0$$

A6. Consider the system

$$\begin{aligned} \dot{x} &= \mu + x^2 + y^2 \\ \dot{y} &= -y + x^2 \end{aligned}$$

(a) State the Center Manifold Theorem for rest points. Briefly explain its importance in bifurcation theory.

(b) Using an approximate center manifold, determine the nature of the bifurcation at the origin at $\mu = 0$.

B. Partial Differential Equations. Do three problems to get full credit

B1. The small longitudinal free vibrations of an elastic bar are governed by the following equation:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[E(x)\frac{\partial u}{\partial x}\right].$$

Here u is the longitudinal displacement, ρ is the linear density of the material and E is its Young's modulus. Assume that the bar is constructed by welding together two bars of different (constant) Young's moduli E_1, E_2 and densities ρ_1, ρ_2, respectively. The displacement u is continuous across the junction, which is located at $x = 0$.

(a) Give a weak formulation of the global initial value problem, and use this to derive the following jump condition:

$$E_1 u_x(0-, t) = E_2 u_x(0+, t), \quad t > 0.$$

(b) Let $c_j^2 = E_j/\rho_j$. A left incoming wave $u_I(x,t) = \mathrm{e}^{i(t-x/c_1)}$ produces at the junction a reflected wave $u_R(x,t) = R\,\mathrm{e}^{i(t+x/c_1)}$ and a transmitted wave $u_T = T\,\mathrm{e}^{i(t-x/c_2)}$. Determine the reflection and transmission coefficients R, T, and interpret the result.

B2. Consider the problem of a thin layer of paint of thickness $h(x,t)$ and speed $u(x,y,t)$ flowing down a wall, see Fig. 1. The paint is assumed to be uniform in the z-direction. The balance

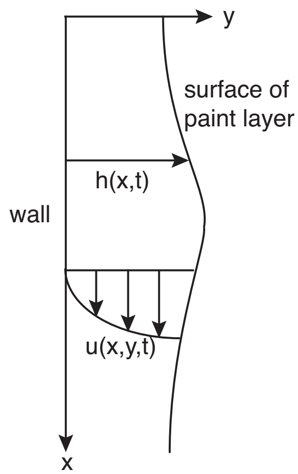

Fig. 1.

between gravity and viscosity (fluid friction) means that the velocity satisfies the equation

$$\frac{\partial^2 u}{\partial y^2} = -c,$$

where c is a positive constant. This is supplemented by the boundary conditions

$$u(x,0,t) = 0, \qquad \left.\frac{\partial u(x,y,t)}{\partial y}\right|_{y=h} = 0.$$

The density of paint per unit length in the x-direction is $\rho_0 h(x,t)$ where ρ_0 is a constant, and the corresponding flux is

$$q(x,t) = \rho_0 \int_0^h u(x,y,t)\,dy.$$

(a) Using conservation of paint, and solving for $u(x,y,t)$ in terms of $h(x,t)$ and y, derive the following PDE for the thickness h:

$$\frac{\partial h}{\partial t} + ch^2 \frac{\partial h}{\partial x} = 0.$$

(b) Set $c = 1$. Show that the characteristics are straight lines and that the Rankine-Hugoniot condition on a shock $x = S(t)$ is

$$\frac{dS}{dt} = \frac{[h^3/3]_-^+}{[h]_-^+}.$$

(c) A stripe of paint is applied at $t = 0$ so that

$$h(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 1, & 0 < x < 1. \end{cases}$$

Show that, for small enough t,

$$h = \begin{cases} 0, & x < 0 \\ (x/t)^{1/2}, & 0 < x < t \\ 1, & t < x < S(t) \\ 0, & S(t) < x, \end{cases}$$

where the shock is $x = S(t) = 1 + t/3$.

(d) Explain why this solution changes at $t = 3/2$, and show that thereafter

$$\frac{dS}{dt} = \frac{S}{3t}.$$

B3. Consider the following inhomogeneous initial-Neumann problem:

$$\begin{aligned} u_t &= Du_{xx} + \alpha t x, && 0 < x < \pi, \quad 0 < t, \\ u(x,0) &= 1, && 0 < x < \pi, \\ u_x(0,t) &= u_x(\pi,t) = 0, && 0 < t; \end{aligned}$$

where α is a constant.

(a) Determine the eigenfunctions of the homogeneous equation

$$u_{xx} = \lambda u, \quad 0 \le x \le \pi, \quad u_x(0) = u_x(\pi) = 0.$$

(b) Solve the inhomogeneous initial-Neumann problem by carrying out an eigenfunction expansion of $u(x,t)$ in terms of the eigenfunctions obtained in part (a). That is, denoting the eigenfunctions by v_k, integer k, set

$$u(x,t) = \sum_{k\ge 0} c_k(t) v_k(x),$$

and determine the time-dependent coefficients $c_k(t)$.

(c) Give a physical interpretation of the solution in the limit $\alpha \to 0$.

B4. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, \quad x \in \mathbf{R}.$$

along with initial conditions

$$u(x,0) = g(x), \qquad u_t(x,0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi.$$

(b) Determine the solution for the initial data $u(x,0) = 1$ if $|x| < a$, $u(x,0) = 0$ if $x > |a|$; $u_t(x,0) = 0$

(c) Determine the solution for the initial data $u(x,0) = 0$; $u_t(x,0) = 1$ if $|x| < a$, $u_t(x,0) = 0$ if $x > |a|$

B5. Suppose that $u(\mathbf{x})$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbf{R}^n$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $\mathbf{x} \in \Omega$ and $r > 0$ is chosen such that $B_r(\mathbf{x}) \subset \Omega$ (ball of radius r centered at $\mathbf{x}$) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) d\mathbf{s},$$

where ω_n is the measure of ∂B_1. Hence show that

$$u(\mathbf{x}) \le \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.