PhD Preliminary Qualifying Examination: Applied Mathematics

Jan. 3, 2013

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. The n data points $(x_i, y_i), i = 1, 2, \ldots, n$ are believed to lie on an exponential curve $y_i = A\exp(\lambda x_i)$. Estimate λ.

2. (a) Specify the weak formulation for the differential equation

$$u'' + \lambda\delta(x - \frac{1}{2})u = f(x)$$

subject to boundary conditions $u(0) = u(1) = 0$.

(b) Find all values of λ for which this differential equation with $f(x) = 0$ has a nontrivial solution and verify that the corresponding solution is a weak solution.

(c) Suppose λ is such that this differential equation with $f(x) = 0$ has only the trivial solution. Find the solution with $f(x) = 1$.

3. (a) Find all conditions on α, β, γ, and $f(x)$ for which solutions of

$$u'' + \alpha^2 u = f(x), \qquad u(0) = \beta, \qquad \alpha u'(0) - u'(1) = \gamma \tag{1}$$

exist.

(b) Find an integral representation for the solution in the case that $\alpha = 0$.

4. Consider the integral equation

$$\phi(x) - \lambda \int_0^\pi \sin(x+t)\phi(t)dt = \cos(x) + \sin(x), \quad 0 \le x \le \pi.$$

(a) Find the unique solution when $\lambda \neq \pm 2/\pi$. Why is a unique solution guaranteed to exist?

(b) Show that there is no solution when $\lambda = 2/\pi$.

(c) Show that when $\lambda = -2/\pi$ there is a one–parameter family of solutions of the form

$$\phi(x) = f_p(x) + \alpha f_h(x).$$

Determine $f_h(x)$.

5. (a) Describe the Gram-Schmidt orthogonalization procedure for a set of n-vectors $\{u_1, u_2, \ldots, u_k\}$.

   (b) Suppose $k < n$ and that the vectors $\{u_1, u_2, \ldots, u_k\}$ are linearly independent. Show that the Gram-Schmidt procedure is equivalent to factoring the matrix U with column vectors $\{u_1, u_2, \ldots, u_k\}$ as $U = QR$ where R is triangular. What is the structure of Q?

   (c) Use this factorization of U to find the least-squares solution of $Ux = b$.

Part B.

1. (a) Formulate and derive the *argument principle* [which determines the difference between the number of zeros (N_0) and poles (N_∞) of an analytic function $f(z)$].

   (b) Applying this principle, determine the number of zeros located inside the first quadrant $\{z = x + iy : \ x > 0, y > 0\}$ of the function $f(z) = z^5 + 1$.

2. Find the image of the half-strip $\{z = x + iy : \quad x > 0, \ 0 < y < 1\}$ under the mapping $w = 1/z$.

3. Calculate the integral

$$I = \int_0^\infty \frac{x^\alpha}{1+x} dx$$

   where α is a real number for which the integral converges.

4. Formulate and derive the *uncertainty principle*. Is its inequality optimal?

5. Find at least three terms of the asymptotic expansion of the integral

$$I(s) = \int_0^1 \ln t \ e^{ist} \, dt, \qquad s \text{ is real and } s \to +\infty.$$