University of Utah, Department of Mathematics
January 2017, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Prove that there is no simple group of order 132.

2. How many elements of the symmetric group S_n commute with a fixed k-cycle in S_n ?

3. Suppose A is a finitely generated abelian group such that $A \otimes_{\mathbb{Z}} \mathbb{F} = 0$ for all finite fields $\mathbb{F}$. Show that $A = 0$.

4. Let G be a finite p-group, for p a prime integer. Let e denote the identity element, and Z the center of G. If $N \neq \{e\}$ is normal, prove that $N \cap Z \neq \{e\}$.

5. Let A be an $m \times n$ real matrix of rank 1. Prove that $A = BC$, for B a column matrix, and C a row matrix.

6. Describe, up to similarity, all 8×8 matrices with minimal polynomial $x^2(x-1)^3$.

7. Let $\mathbb{F}_5$ be the field with 5 elements. Construct an explicit isomorphism between the fields $\mathbb{F}_5[x]/(x^2-2)$ and $\mathbb{F}_5[x]/(x^2-3)$.

8. Let $K = \mathbb{Q}(\pi)$, where π is the ratio of the circumference of a circle to its diameter. You may assume without proof that π is transcendental over $\mathbb{Q}$.

   (a) Show that $f(x) = x^3 - \pi$ is irreducible over K.

   (b) Let L denote the splitting field of $f(x)$ over K. Compute $[L : K]$ and $\mathrm{Gal}(L/K)$.

9. Let a be an integer and p a prime number such that $x^p - a$ is reducible in $\mathbb{Q}[x]$. Prove that $x^p - a$ has root in $\mathbb{Q}$.

10. Let a, b, c, d integers such that

$$\det(A) \neq 0 \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   Let x and y be indeterminates, and set $L = \mathbb{C}(x, y)$ and $K = \mathbb{C}(x^a y^b, x^c y^d)$.

   (a) Prove that L is a finite extension of K and that $(L : K) = |\det(A)|$.

   (b) Show that L/K is Galois, and compute the Galois group.