ALGEBRA EXAM - JANUARY 2008

(1) Show that a finite group of order 105 has a non-trivial normal subgroup.

(2) Let p be a prime. Determine all groups of order p^2.

(3) Let R be a commutative ring with 1. Assume that R satisfies the ascending chain condition. Let I be an ideal generated by an infinite sequence of elements x_1, $x_2, \ldots$ in R. Show that I is finitely generated.

(4) Let $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, a cube root of 1. Show that the ring $\mathbb{Z}[\sqrt{\omega}]$ is a euclidean domain.

(5) Let A be a rational 3×3 matrix such that $A^3 = A$. Show that A can be diagonalized.

(6) Describe all finitely generated $\mathbb{Z}$-submodules of $\mathbb{Q}$.

(7) Show that $1 \otimes (1, 1, \ldots) \neq 0$ in the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n=2}^{\infty} (\mathbb{Z}/n\mathbb{Z})$.

(8) Prove that $\Phi_{2^n}(x) = x^{2^{n-1}} + 1$ is irreducible in $\mathbb{Q}[x]$.

(9) Find a real number α such that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ with the Galois group $\mathbb{Z}/5\mathbb{Z}$.

(10) Let G be the group of symmetries of an equilateral triangle. Show that the 2 dimensional representation (as symmetries of the triangle) is irreducible by calculating the character table.