Math 5210, Topics for the Final Exam

The final exam will be comprehensive, but with more emphasis on the more recent topics. I will ask questions from the material already covered in the first two midterms, as well as in the new material, roughly half old and half new. References are: N = notes, P = Pugh, chapter 6, R = Rudin.

1. Basic material on metric spaces: Definitions, examples, Cauchy sequences, completeness.
2. Inequalities: Cauchy Schwarz and related ones, Jensen’s inequality (convexity) (N4).
3. Normed spaces: \( \mathbb{R}^n \) and \( \mathbb{R}^\infty \), \( C[a,b] \) with the \( p \)-norms, \( 1 \leq p \leq \infty \). (N4)
4. Completeness of \( C[a,b] \), \( ||f||_p \) for \( p = \infty \), incompleteness for \( 1 \leq p < \infty \). (N3).
5. Contraction mapping theorem and applications. (R9, N5)
6. Basic material on differentiable maps from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) (R9).
7. Inverse and implicit function theorems (R9, N6).
8. Spaces of continuous functions:
   (a) Equicontinuous families, compact subsets of \( C(X) \) for \( X \) compact metric (R7.22 to 7.25).
   (b) Arzela - Ascoli theorem: \( \{f_n\} \) a bounded, equicontinuous sequence in \( C(X) \), has a uniformly convergent subsequence. (R7.25).
   (c) Weierstrass approximation theorem: Polynomials are dense in \( C[0,1] \). (R7.26)
9. Lebesgue Theory: Definitions
   (a) Lebesgue outer measure \( m^* \) in \( \mathbb{R} \) and \( \mathbb{R}^2 \), sets of measure zero. (P1):
      \[
      m^*(E) = \inf \{ \sum_i |I_i| : \{I_i\} \text{ countable collection of open intervals with } \bigcup I_i \supseteq E \}
      \]
   (b) Abstract outer measure \( \omega : 2^M \to [0,\infty] \), where \( M \) is any set: a monotone, countably subadditive function with \( \omega(\emptyset) = 0 \) (P2).
   (c) Sigma algebra: a collection of subsets of \( M \) containing \( \emptyset \) and closed under complement and countable union. Measure on a sigma-algebra: a monotone, countably additive \([0,\infty]-valued function on the sigma algebra, assigning 0 to \( \emptyset \). (P2)
   (d) Given a set \( M \) and an abstract outer measure on \( 2^M \), definition of measurable sets and of the measure of a measurable set. (P2) A set \( E \subset M \) is measurable if and only if, for all \( X \subset M \),
      \[
      \omega(X) = \omega(X \cap E) + \omega(X \cap E^c).
      \]
      If \( E \) is measurable, its measure is its outer measure.
(e) In $\mathbb{R}$ or $\mathbb{R}^2$: a $G_\delta$-set is a countable intersection of open sets, an $F_\sigma$-set is a countable union of closed sets (P3).

(f) (P4) For any function $f : \mathbb{R} \to [0, \infty)$, define its undergraph

$$\mathcal{U}(f) = \{(x, y) \in \mathbb{R}^2 : 0 \leq y < f(x)\}$$

Then

i. $f$ is **Lebesgue measurable** if and only if $\mathcal{U}(f)$ is a Lebesgue measurable subset of $\mathbb{R}^2$.

ii. If $f$ is Lebesgue measurable, its Lebesgue integral is defined to be

$$\int f = m_2(\mathcal{U}(f))$$

where $m_2$ denotes Lebesgue measure in $\mathbb{R}^2$ (two-dimensional Lebesgue measure).

(g) (P5) Upper and lower Lebesgue sums for a partition $Y$ of $[0, \infty)$, $0 = y_0 < y_1 < y_2 < \ldots \to \infty$:  

$$L(f, Y) = \sum_{i=1}^{\infty} y_{i-1}m(X_i), \quad L(f, Y) = \sum_{i=1}^{\infty} y_im(X_i),$$

where $X_i = f^{-1}([y_{i-1}, y_i))$.


(a) Countable subsets of $\mathbb{R}$ or $\mathbb{R}^2$ have measure zero (P1).

(b) Lebesgue outer measure of an interval is its length, of a rectangle is its area. (P1)

(c) Limit properties of measures: upward and downward measure continuity theorems (P2): If $E_n$ are measurable, and $E_n \uparrow E$, then $E$ is measurable and $\omega(E_n) \uparrow \omega(E)$. If $E_n \downarrow E$ and $\omega(E_1) < \infty$, then $E$ is measurable and $\omega(E_n) \downarrow \omega(E)$.

(d) In $\mathbb{R}$ intervals are measurable, sets of measure zero are measurable (P3).

(e) Regularity of Lebesgue measure: if $E$ is measurable set in $\mathbb{R}$ or $\mathbb{R}^2$, there exist a $G_\delta$ set $G$, an $F_\sigma$-set $F$ so that $F \subset E \subset G$ and $m(G \setminus F) = 0$ (P3).

(f) Convergence Theorems: Monotone Convergence, Dominated Convergence (P4). Know how these can fail for Riemann integrals (homework).

(g) Fatou’s Lemma (P4). Know examples that show inequality can be strict (homework).

(h) If $f$ is Lebesgue integrable, then $L(f, Y) \uparrow \int f$, as mesh of $Y \to 0$ (P5).