New problem 1

Let us consider a homogeneous linear system with unknowns $x_1, \ldots, x_n$.

There is a theorem, that we proved in the lecture, that a solution set of a linear system is equal to

$$ A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_0 + x_1 q_1 + \ldots + x_n q_n : x_0, \ldots, x_n \in \mathbb{R} \right\} \quad (1) $$

$q_1, \ldots, q_n$ are linearly independent\(^{2}\) and \( k \) is the number of free variables \(^{3}\)

In the case of a homogeneous system \( q_0 = 0 \).

In this case, using (1) one obtains:

$$ A = \text{span} \left( q_1, \ldots, q_k \right) \quad (4) $$

As \( \text{span} \) is a subspace, then \( A \) is a subspace of \( \mathbb{R}^n \).

As (4) and (2) is true, \( \left\{ q_1, \ldots, q_k \right\} \) is a basis of \( A \). Thus

$$ \dim(A) = k = \text{the number of free variables} \quad (3) $$

Example

Let the linear system have an RREF of the augmented matrix

$$ \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 3 & 4 & 0 \end{bmatrix} $$

The solution set is

$$ A = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ -4 \\ 1 \end{bmatrix} : x_0, x_1, x_2 \in \mathbb{R} \right\} = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right) = \text{span}(q_1, q_2) $$. 

$$ \dim(A) = k = 2 $$

One can use this problem to find the dimension of some of the subspaces of problem 4.

\( b \) \( W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{R}, x_1 + x_2 = 0 \right\} \) — this is a solution set of a homogeneous linear system with augmented matrix \( \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \). It has 1 free variable

So \( \dim(W) = 1 \). Other method would be to find a basis of \( W \), which is \( \{ e_1 \} \)