Linear Algebra 2270
Homework 10
preparation for the quiz on 07/29/2015

Problems:

1. Find the rank, the inverse, and the \( LU \) decomposition of the following matrices. For the \( LU \) decomposition, DO NOT transform matrix \([A|I]\) into \([U|L]\). Instead transform matrix \( A \) to a matrix \( U \) in REF and use the coefficients of the row replacement operations to form \( L \).

(a) 
\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 2 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

(b) 
\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

(c) 
\[
A = \begin{bmatrix}
1 & 1 \\
1 & 0 \\
1 & 0
\end{bmatrix}
\]

Hint: (b) and (c) are tricky.

2. Is there a \( 2 \times 3 \) matrix with rank 3?

3. Find the cosine of the angle between the following two vectors:
\[
v_1 = \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
-3 \\
0 \\
4
\end{bmatrix}
\]

4. Find an example of a \( 3 \times 2 \) matrix \( A \) such that 
\[
A^T A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I
\]

Notice that \( A \) is not square, so \( A^{-1} \) does not exist. Compare it with M8).

5. Show that the following is an inner product (it satisfies I1) – I4).
\[
\text{for } x, y \in \mathbb{R}^2, \quad (x, y) = y^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x
\]

6. Show that the following is an inner product (it satisfies I1) – I4).
\[
\text{for } x, y \in \mathbb{R}^2, \quad (x, y) = y^T \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} x
\]

Hint: to prove I1), you might want to use the fact that for \( a, b \in \mathbb{R}, \quad (a + b)^2 = a^2 + 2ab + b^2 \)

7. Show that the following is NOT an inner product.
\[
\text{for } x, y \in \mathbb{R}^2, \quad (x, y) = y^T \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} x
\]
8. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Show that the following is an inner product in $\mathbb{R}^n$:

$$\text{for } x, y \in \mathbb{R}^n, \quad (x, y) = y^T (A^T A)x = (Ay)^T (Ax)$$

9. Let $(.,.)$ be an inner product. Prove that

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

If $(.,.)$ is the dot product in $\mathbb{R}^2$, there is an interpretation of this formula involving sides and diagonals of a parallelogram. What is the interpretation?

10. Let $V$ be an inner product space. Let the norm be defined using the inner product:

$$\|x\| = \sqrt{(x, x)}$$

Prove that for any $x, y \in V$

$$(x, y) = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

The formula above implies that the values of the inner product are completely determined by the values of the norm. Or in other words, if you know the values $(x, x)$ for all $x \in V$, then you know the values $(x, y)$ for all $x, y \in V$. 

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