Some additional definitions needed for problems 5 and 6: Two linear systems are called equivalent if they have the same solution set. A matrix with \( m \) rows and \( n \) columns is called an \( m \times n \) matrix. A set of all real \( m \times n \) matrices is denoted as \( \mathbb{R}^{m \times n} \).

Problems:

1. At some store one candy costs $0.5 and one chocolate bar costs $1. The sales tax one has to pay for one candy is $0.05. The sales tax paid for one chocolate bar is $0.15.

   (a) How much would 10 candies and 4 chocolate bars cost? How much sales tax would one have to pay?

   (b) How many chocolate bars and candies Tom bought if he paid $9 for the sweets plus $1.25 for the sales tax?

2. Solve the following linear system, using whatever method you wish:

   \[
   \begin{align*}
   x_1 + 5x_2 &= 7 \\
   -2x_1 - 7x_2 &= -5
   \end{align*}
   \]

3. Find the point \((x_1, x_2)\) that lies on the line \( x_1 + 2x_2 = 4 \) and on the line \( x_1 - x_2 = 1 \). See the figure.

   \[
   \begin{align*}
   x_1 + x_2 &= 1 \\
   x_1 + 2x_2 &= 4
   \end{align*}
   \]

   Hint: What linear system do the coordinates \((x_1, x_2)\) satisfy?

4. A good way to solve a linear system with \( n \) equations and \( n \) unknowns is to apply elementary row operations to transform the augmented matrix into a matrix with “1” on the diagonal and zeros above and below the diagonal, and some number in the last columns. For a system with three unknowns and three equations, the desired form of the matrix would be:

   \[
   \begin{bmatrix}
   1 & 0 & 0 & | & \ast \\
   0 & 1 & 0 & | & \ast \\
   0 & 0 & 1 & | & \ast
   \end{bmatrix}
   \]

   where “\( \ast \)” denotes an arbitrary number. Such a matrix corresponds to a linear system that may be written as

   \[
   \begin{cases}
   x_1 = \ast \\
   x_2 = \ast \\
   x_3 = \ast
   \end{cases}
   \]

   for which the solution is obvious.

   Using this technique solve the following two linear systems:
Perform one operation at a time and use the following notation for row operations:

- (replacement) \( A r_k + c r_i \rightarrow B \) denotes the operation in which the \( i \)-th row is multiplied by \( c \) and added to the \( k \)-th row.
- (scaling) \( A r_k = c r_k \rightarrow B \) denotes the operation in which the \( k \)-th row is multiplied by \( c \).
- (interchange) \( A r_k \leftrightarrow r_i \rightarrow B \) denotes the operation in which the \( k \)-th row is interchanged with the \( i \)-th row.

For example:

\[
\begin{bmatrix}
1 & 2 & 1 \\
-1 & 1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\xrightarrow{r_1 + (-1)r_3}
\begin{bmatrix}
0 & 2 & -1 \\
-1 & 1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\xrightarrow{r_2 = 3 r_2}
\begin{bmatrix}
0 & 6 & -3 \\
-1 & 1 & 1 \\
1 & 0 & 2
\end{bmatrix}
\xrightarrow{r_2 \leftrightarrow r_3}
\begin{bmatrix}
0 & 6 & -3 \\
1 & 0 & 2 \\
-1 & 1 & 1
\end{bmatrix}
\]

5. Determine which of the following sentences are true and which are false:

a. Every elementary row operation is reversible.

b. A \( 5 \times 6 \) matrix has six rows.

c. The solution set of a linear system involving variables \( x_1, \ldots, x_n \) is a list of numbers \( (s_1, \ldots, s_n) \) that makes each equation in the system a true statement when the values \( s_1, \ldots, s_n \) are substituted for \( x_1, \ldots, x_n \), respectively.

d. Two fundamental questions about a linear system involve existence and uniqueness.

6. Determine which of the following sentences are true and which are false:

a. Two matrices are row equivalent if they have the same number of rows.

b. Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

c. Two equivalent linear systems can have different solution sets.

d. A consistent system of linear equations has one or more solutions.

7. Determine if the following system is consistent. Do not completely solve the system:

\[
\begin{align*}
x_2 + 5x_3 &= -4 \\
x_1 + 4x_2 + 3x_3 &= -2 \\
2x_1 + 7x_2 + x_3 &= -2 \\
x_1 - 5x_2 + 4x_3 &= -3 \\
2x_1 - 7x_2 + 3x_3 &= -2 \\
-2x_1 + x_2 + 7x_3 &= -1
\end{align*}
\]
\begin{align*}
x_1 - 6x_2 &= 5 \\
x_2 - 4x_3 + x_4 &= 0 \\
-x_1 + 6x_2 + x_3 + 5x_4 &= 3 \\
-x_2 + 5x_3 + 4x_4 &= 0
\end{align*}