Sample Problems for MidTerm 1

1. Does the superposition principle hold for the following (take each item separately):
   (a) PDE \( u_{tt} + u_{xx} + u^3 = 0 \)
   (b) PDE \( u_{tt} + u_{xx} + x^3 = 0 \)
   (c) PDE \( u_{tt} + x^3 u_{xx} = 0 \)
   (d) boundary condition \( u_x(1, t) = 2u(1, x) \)
   (e) boundary condition \( u_x(0, t) = 2u(0, x) \)
   (f) boundary condition \( u(0, t) = 2 \)
   (g) boundary condition \( u(1, t) = 2 \)
   (h) ODE \( u'' = u' + u \)
   (i) ODE \( u'' = uu' \)
   (j) PDE \( u_x \cos x + u_y \sin y = 0 \)
   (k) ODE \( \frac{du}{dt} \cos t + u \sin t = 0 \)
   (l) PDE \( u_{tt} = |u|_{xx} ? \)

   Explain.

2. (a) Find the general solution \( u(x, t) \) of the following equation

   \[ u_t + (t - 1)u_x = 0 \quad (-\infty < x < \infty, -\infty < t < \infty). \]

   (b) Find the solution of this equation satisfying initial condition

   \[ u(x, 0) = \frac{1}{1 + x^4}. \]

   (c) Schematically plot the snapshots of this solution for \( t = 0, 1, 2 \).

3. (a) Let \( F \) and \( G \) be arbitrary (smooth) functions of one variable. Show that \( u(x, t) = F(x + ct) + G(x - ct) \) is a solution of the wave equation \( u_{tt} = c^2 u_{xx} \) \( (-\infty < x < \infty, t > 0) \).

   (b) Solve the wave equation with initial conditions

   \[ u(x, 0) = e^{-x^2}, \quad u_t(x, 0) = 0. \]

   (c) Schematically plot the snapshots of this solution for \( t = 0, 10, 20 \).

4. Find the Fourier series for the following functions on the interval \((-\pi, \pi)\):

   (a) \[ f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ +1 & \text{if } 0 < x < +\pi. \end{cases} \]
   (b) \[ g(x) = 1 + (\sin x)(\cos x) - \cos x; \]
   (c) \[ h(x) = f(x) + g(x). \]

   Write a 3-term Fourier expansion in each case (i.e. an expansion consisting of 3 lowest non-zero terms).

5. Consider an electric circuit modelled by the differential equation

   \[ y'' + 0.02y' + 8.99y = f(t) \]

   where input \( f(t) \) is a \( 2\pi \)-periodic function such that \( f(t) = 1 \) if \( 0 < t < \pi \) and \( f(t) = -1 \) if \( -\pi < t < 0 \).

   (a) Show that the general solution of the homogeneous equation "dies out" as \( t \to \infty \).

   (b) Explain why the system "forgets" initial conditions.

   (c) Determine the dominant term in the response \( y(t) \).