Sketch of solutions for the homework assignment 1.

Ex 2.1 #8 \( f(x) = \cos(x) + \cos(\pi x) \).
(a) By remarking that \( \cos(x) \leq 1 \) and \( \cos(\pi x) \leq 1 \) for all \( x \), we have that

\[
\begin{align*}
f(x) = 2 & \iff \cos x + \cos(\pi x) = 2 \\
& \iff \left\{ \begin{array}{l}
\cos x = 1 \\
\cos(\pi x) = 1
\end{array} \right.
\end{align*}
\]

The latter system has the following solutions: \( x = 2k\pi \) for any integer \( k \) and \( x = 2l \) for any integer \( l \). The only value of \( k \) and \( l \) where those two coincide is when \( l = k = 0 \) that is \( x = 0 \), because of the irrationality of \( \pi \) (no integer multiple of \( \pi \) can be an integer).

This proves that the only solution to the equation \( f(x) = 2 \) is 0.
(b) Let’s assume that \( f \) was periodic of period \( T \). We will prove that this is impossible by finding a contradiction.

We know that \( f(0) = 2 \), then because \( f(T) = f(0) \) (\( T \)-periodicity) then we also have \( f(T) = 2 \), meaning that we would have another solution to the equation \( f(x) = 2 \) which is impossible because of (a).

Hence the contradiction, then \( f \) is not periodic.

Ex 2.1 #15 You have to prove two things: if \( F \) is \( 2\pi \)-periodic then \( \int_0^{2\pi} f(t)dt = 0 \) and the converse.

First let us assume that \( F \) is \( 2\pi \)-periodic and then prove that \( \int_0^{2\pi} f(t)dt = 0 \). Because \( F \) is \( 2\pi \)-periodic we have that \( F(0) = F(2\pi) \) then this is

\[
\begin{align*}
\int_a^0 f(t)dt &= \int_a^{2\pi} f(t)dt \Rightarrow - \int_0^a f(t)dt + \int_a^{2\pi} f(t)dt = 0 \\
& \Rightarrow \int_0^a f(t)dt + \int_a^{2\pi} f(t)dt = 0 \Rightarrow \int_0^{2\pi} f(t)dt = 0
\end{align*}
\]

To prove the converse: assume that \( \int_0^{2\pi} f(t)dt = 0 \). We want to prove that \( F(x + 2\pi) = F(x) \) for any \( x \). We calculate \( F(x + 2\pi) - F(x) \):

\[
F(x + 2\pi) - F(x) = \int_a^{x + 2\pi} f(t)dt - \int_a^x f(t)dt = \int_a^{x + 2\pi} f(t)dt + \int_x^{x + 2\pi} f(t)dt
\]

\[
= \int_x^{x + 2\pi} f(t)dt
\]
but as $f$ is $2\pi$-periodic we have (theorem 1) that

$$
\int_{x}^{x+2\pi} f(t) dt = \int_{0}^{2\pi} f(t) dt = 0
$$

then we have proved $F(x + 2\pi) - F(x) = 0$ which means that

$$
F(x + 2\pi) = F(x)
$$

2.1 #19 The case $p = 1$ (the other cases are just the same with a scale factor).
You have to break the study on intervals of length 1.
For example for $x \in [0, 1)$ we have that $[x] = 0$ then $f(x) = x$.
For $x \in [1, 2)$ we have that $[x] = 1$ then $f(x) = x - 1$.
More generally for $x \in [n, n+1)$ we have that $[x] = n$ then $f(x) = x - n$,
where $n$ is any integer.
Then the graph of $f$ is
It is obvious on the graph that the period should be 1, still we must

prove it: take any $x$; remarking that $[x + 1] = [x] + 1$ we have

$$
\begin{align*}
    f(x + 1) &= x + 1 - [x + 1] = x + 1 - ([x] + 1) = x - [x] = f(x)
\end{align*}
$$

then $f$ is 1-periodic (it is clear that 1 is the period).