3.3.10 Let’s first find vectors in the kernel by finding linear relations among the columns:
\[
\begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix}
\]
It does not seem to have another one. We will calculate the dimensions of the image and the kernel. First we remark that
\[
\text{Im } A = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}
\]
But those 2 vectors are linearly independent (just row reduce the corresponding matrix, or use the definition), so they form a basis of \text{Im } A, then \text{rank } A = \text{dim Im } A = 2. But if we apply the rank-nullity theorem we have
\[
\text{rank } A + \text{dim ker } A = 3 \implies \text{dim ker } A = 1
\]
But then as \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0 \) is in the kernel, it forms a basis (indeed a set of vectors consisting of only one non-zero vector is always linear independent).

3.3.14 Here we can just remark that \text{rank } A = 1 because it is already reduced. Then using the rank-nullity theorem we have
\[
\text{rank } A + \text{dim ker } A = 3 \implies \text{dim ker } A = 2
\]
Moreover as \text{dim Im } A = \text{rank } A = 1 we need only one non-zero vector to form a basis, then we have (1) which forms a basis of \text{Im } A.
Finding linear relations among the columns we find the following vectors in the kernel:
\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}
\]
Those two vectors are linearly independent because
\[
\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \implies \text{rank } \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} = 2
\]
Then those two vectors form a basis of \text{ker } A.

3.3.16 We row-reduce it to find its rank:
\[
A = \begin{pmatrix}
1 & 1 & 5 & 1 \\
0 & 1 & 2 & 2 \\
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 4 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 3 & -1 \\
0 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
then rank $A = 3$. Then using the rank-nullity theorem we have

$$\dim \ker A + \text{rank } A = 4 \implies \dim \ker A = 1$$

$$\text{Im } A = \text{span}\left\{\left(\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \\ 3 \\ 4 \end{array}\right), \left(\begin{array}{c} 1 \\ 2 \\ 1 \\ 3 \\ 4 \end{array}\right)\right\}$$

As a side note, we know that \(\text{rank } A = \dim \text{Im } A\) but \(\text{rank } A \leq 2\) so we deduce that we have at most two linear independent vectors among those spanning vectors.

We can build the following linear relations:

$$-\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right) + 2\left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right) = 0 \quad \text{and} \quad -\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right) + \left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right) - \left(\begin{array}{c} 1 \\ 4 \end{array}\right) = 0$$

then we deduce that

$$-\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right) + 2\left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right) = \left(\begin{array}{c} 1 \\ 3 \end{array}\right) \quad \text{and} \quad -\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right) + \left(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right) = \left(\begin{array}{c} 1 \\ 4 \end{array}\right)$$

so the last two vectors are redundant, we can discard them that is we have

$$\text{Im } A = \text{span}\left\{\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right), \left(\begin{array}{c} 1 \\ 2 \end{array}\right)\right\}$$

To check that we cannot go further we prove that those two vectors are linearly independent. We do it by finding the rank of the following matrix

$$B = \left(\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right) \det B = 2 - 1 = 1 \neq 0 \implies B \text{ is invertible} \implies \text{rank } B = 2$$

then the vectors \(\left(\begin{array}{c} 1 \\ 1 \\ 2 \end{array}\right), \left(\begin{array}{c} 1 \\ 2 \end{array}\right)\) are linearly independent.

3.3.28 First we have 4 vectors in \(\mathbb{R}^4\), so they will form a basis if and only if they are linearly independent as \(\dim \mathbb{R}^4 = 4\). So we need to check for which values of \(k\) those vectors are independent. We build a matrix (4x4) which columns are those vectors, and find its rank:

$$A = \left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 4 & k - 4 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k - 29 \end{array}\right)$$

So if \(k = 29\), the last line is full of zeros and then rank \(A < 4\) then the vectors are linearly dependent. Moreover if \(k \neq 29\) then the reduction can be carried on to its end

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k - 29 \end{array}\right) \sim \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

and then rank \(A = 4\) and then the vectors are linearly independent. So those 4 vectors form a basis of \(\mathbb{R}^4\) if and only if \(k \neq 29\).
3.4.7 We have \[ x = 3v_1 + 4v_2 \]
so \( x \in \text{span}\{v_1, v_2\} \) and \([x]_B = \left( \begin{array}{c} 3 \\ 2 \end{array} \right)\).

3.4.9 There is no obvious relation so we use the row-reduced form to decide:
\[
A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 1 & -1 \\ 4 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 0 & -1 \\ 0 & -4/3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
then \( \text{rank } A = 3 \) and then \( x, v_1, v_2 \) are linearly independent therefore \( x \notin \text{span}\{v_1, v_2\} \).

3.4.21 First we have
\[
S = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}
\]
First remark that \( \det S = 1 + 6 = 7 \neq 0 \), then \( S \) is invertible, which proves that \( B \). Now we have
\[
S^{-1} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}
\]
Then we can compute \( B \):
\[
B = S^{-1}AS = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 21 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 0 \end{pmatrix}
\]
Another way to see that is to calculate \( T(v_1), T(v_2) \) in the \( B \)-coordinates:
\[
\begin{align*}
T(v_1) &= \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix} = 7v_1 \implies [T(v_1)]_B = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \\
T(v_2) &= \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies [T(v_2)]_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\]

3.4.23 First we have
\[
S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}
\]
First remark that \( \det S = 2 - 1 = 1 \neq 0 \), then \( S \) is invertible, which proves that \( B \). Now we have
\[
S^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}
\]
Then we can compute \( B \):
\[
B = S^{-1}AS = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}
\]
Another way to see that is to calculate \( T(v_1), T(v_2) \) in the \( B \)-coordinates:
\[
\begin{align*}
T(v_1) &= \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2v_1 \implies [T(v_1)]_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
T(v_2) &= \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -v_2 \implies [T(v_2)]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\end{align*}
\]