Exercise 4: First recall that \( \mathbb{C} \) is \( \mathbb{R}^2 \). That is a complex number can be seen as a vector in \( \mathbb{R}^2 \) in the following way:

\[ \mathbb{C}, \text{ as viewed as the vector } (\Re(z), \Im(z)) \in \mathbb{R}^2 \] (real and imaginary parts)

So the vector \((a, b)\) \(\in \mathbb{R}^2\) can be seen, conversely, as the complex number \(a + ib \in \mathbb{C}\) (i is the root of \(-1\) : \(i^2 = -1\)).

So \( \mathbb{C} \to \mathbb{C} \) is then also a map from \( \mathbb{R}^2 \to \mathbb{R}^2 \).

Moreover, the operations + and scaling in \( \mathbb{R}^2 \) correspond to the addition of complex numbers and multiplication of a complex number by a real number:

- in \( \mathbb{C} \):
  \[ z = a + ib \in \mathbb{C}, \quad z = a + ib \in \mathbb{C} \]
  \[ \text{if } z = x + iy, \quad \text{then } z' = x' + iy' \]
  \[ z + z' : \text{complex numbers addition} \]
  \[ z \cdot a : \text{multiplication of } z \text{ by the real number} \]

Now we can use the two ways of viewing \( \mathbb{C} \) to prove that \( T \) is a linear map:

\[
\begin{align*}
\forall z, z' \in \mathbb{C}, \forall a \in \mathbb{R}, \quad & J(z + z') = \frac{z + z'}{i} = \frac{i(z + z')}{i^2} = J(z) + J(z') \\
J(a z) = \frac{a z}{i} = \frac{a z}{i^2} = a \frac{z}{i} = a J(z), & \text{because } a \in \mathbb{R}
\end{align*}
\]