The Grothendieck Ring of Varieties

Ziwen Zhu

University of Utah

May 21, 2018

These are supposed to be the notes for a talk of the student seminar in algebraic geometry.

In the talk, We will first define the Grothendieck ring of varieties and discuss some basic facts about it. Then we will see how it can be used to study birational geometry. The base field will be $\mathbb{C}$ throughout the talk for simplicity although some of the results actuallly work for arbitrary base field. Most of the materials in this note are discussed in more detail in [GS14].

1 Definition and Basic Properties

1.1 Defnition of the Grothendieck ring of varieties

We denote by $K_0(Var/\mathbb{C})$ the Grothendieck ring of varieties over $\mathbb{C}$. As an abelian group, it is generated by isomorphism classes of quasi-projective varieties over $\mathbb{C}$ modulo the following relation:

$$X = U + Z,$$

if $U \subset X$ is open and $Z = X \backslash U$. We denote by $[X]$ the class of X in $K_0(Var/\mathbb{C})$. The product on $K_0(Var/\mathbb{C})$ is defined to be

$$[X] \cdot [Y] = [X \times Y].$$

Note that the product is well defined. Indeed, if $[X] = [U] + [Z]$, then $X \times Y$ is the disjoint union of $U \times Y$ and $Z \times Y$, where $Z \times Y$ is open inside $X \times Y$. Therefore, $[X \times Y] = [U \times Y] + [Z \times Y]$.

Remark. The following is immediate from the definition:

1. The zero element in $K_0(Var/\mathbb{C})$ is the class of the empty set $[\varnothing]$, and the identity in $K_0(Var/\mathbb{C})$ is the class of a point $[pt]$.

2. We use $\mathbb{L}$ to denote the class of $\mathbb{A}^1$.

3. The addition in the Grothendick ring can be considered as disjoint union. Elements in $K_0(Var/\mathbb{C})$ can be written as the sum of its disjoint locally closed subvarieties.

1.2 Basic properties

We can also define symmetric products on the Grothendieck ring. First of all, recall for a variety X, we define its n-th symmetric power as $X^{(n)} = Sym^n(X) := X^n/S_n$, where S_n is the n-th symmetric group. We can pass this definition to the Grothendieck ring as well by defining $[X]^{(n)} = [X^{(n)}]$. Indeed, if $[X] = [U] + [Z]$, then $X^{(n)}$ can be decomposed into $U^{(i)} \cdot Z^{(j)}$ for $i + j = n$. In particular, for any two classes α and β in $K_0(Var/\mathbb{C})$, we have

$$(\alpha + \beta)^{(n)} = \sum_{i+j=n} \alpha^{(i)} \cdot \beta^{(j)}.$$

We have examples of two nonisomorphic varieties X and X' such that their difference $[X] - [X'] = 0$ in $K_0(Var/\mathbb{C})$. In fact, if there is a bijective morphism $f : X' \to X$, the degree of f, which is $[K(X') : K(X)]$, is going to be the number of points in a general fiber, which is 1 in this case. Hence X and X' are birational. Subtracting the isomorphic open subsets inside X and X', we can go on by induction on the dimension of X. Here is a concrete example of this: Let C be the cusp $(y^2 - x^3 = 0)$ in $\mathbb{A}^2$, $\widetilde{C}$ be its normalization, then $\widetilde{C} \to C$ is a bijective morphism and $[C] = [\widetilde{C}]$.

Some flips $X \dashrightarrow X^+$ also have the property that $[X] = [X^+]$. For example, the flip resulting by blowing up of the cone over the quadric in $\mathbb{P}^3$.

There are two very useful formulas in the Grothendieck ring, one for a fibration and one for blowing up along a smooth center.

Let $\pi : X \to S$ be a Zariski locally-trivial fibration (or in other words a fiber bundle in the Zariski topology) with fiber F. Then

$$[X] = [F] \cdot [S]. \tag{1.1}$$

We can prove this by induction on the dimension of S. First of all, if $\dim S$ is 0, then S is a disjoint union of finitely many points, and apparently $[X] = [F] \cdot [S]$. Now for S with larger dimension, because the fibration $X \to S$ is locally trivial, we can find an open subset U inside S such that $\pi^{-1}(U) = U \times F$. Therefore, we can write

$$[X] = [U] \cdot [F] + \pi^{-1}(S\backslash U),$$

where $\pi : \pi^{-1}(S\backslash U) \to S\backslash U$ is a locally trivial fibration over $S\backslash U$, which is of smaller dimension. Therefore, by induction hypothesis, $\pi^{-1}(S\backslash U) = [S\backslash U] \cdot [F]$, and

$$[X] = [U] \cdot [F] + [S\backslash U] \cdot [F] = [S] \cdot [F].$$

Let X be a smooth variety and $Z \subset X$ a smooth closed subvariety of codimension c. We denote by $Bl_Z(X)$ the blowing up of X along Z. Then

$$[Bl_Z(X)] = [X] - [Z] + [\mathbb{P}^{c-1}][Z].$$

This follows directly from the definition if we notice that the exceptional divisor $[E] = [\mathbb{P}(\mathcal{N}_{Z/X})]$ is a $\mathbb{P}^{c-1}$ bundle over Z, where $\mathcal{N}_{Z/X}$ is the normal bundle of Z inside X.

2 Grothendieck Ring's Relation to Birational Geometry

We can use the Grothendieck ring to study rationality problems.

Proposition 2.1. Let X, X' be smooth birationally equivalent varieties of dimension d. Then we have the following equality in the Grothendieck ring $K_0(Var/\mathbb{C})$:

$$[X'] - [X] = \mathbb{L} \cdot \mathcal{M},$$

where $\mathcal{M}$ is a linear combination of classes of smooth projective varieties of dimension $\leq d-2$.

Proof. We need the Weak Factorization Theorem [Wło03,AKMW02]. It says that a birational map between two proper smooth varieties can be factored as a series of birational maps, where each birational map is either a blow-up or blow-down along a smooth center. Therefore here we may assume $X' \to X$ is the blow-up of X along Z, with $\operatorname{codim}(Z, X) = c \geq 2$. Now by the formula we deduced previously, we have

$$[X'] - [X] = ([\mathbb{P}^{c-1}] - 1) \cdot [Z] = (\sum_{n=1}^{c-1} \mathbb{A}^n) \cdot [Z] = \mathbb{L} \cdot (\sum_{n=0}^{c-2} \mathbb{A}^n) \cdot [Z] = \mathbb{L} \cdot [\mathbb{P}^{c-1} \times Z].$$

□

And the dimension of $\mathbb{P}^{c-1} \times Z$ is $\leq d-2$.

Immediately, we get a corollary which gives a necessary condition for a smooth variety to be rational.

Corollary 2.2. If X is a rational smooth variety of dimension d, then

$$[X] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_X,$$

where $\mathcal{M}_X$ is a linear combination of classes of smooth projective varieties of dimension $\leq d-2$.

This leads to the following definition:

Definition 2.3. Let X be a variety of dimension d. We call the class

$$\mathcal{M}_X := \frac{[X] - [\mathbb{P}^d]}{\mathbb{L}} \in K_0(Var/\mathbb{C})[\mathbb{L}^{-1}]$$

the rational defect of X.

The Corollary 2.2 tells us that if X is smooth and rational, then the rational defect $\mathcal{M}_X$ can be taken to be an element in $K_0(Var/\mathbb{C})$.

Next we will list some more advanced questions and results about the Grothendieck ring that are related to birational geometry.

First of all, it is still open whether $[X] = [Y]$ implies that X and Y are birationally equivalent. However, this does imply X is stably birational to Y. Recall that two smooth varieties X and Y are called stably birationally equivalent if for some $m, n \geq 1$, $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$. In fact, we have the following stronger result, which is also a weaker version of the converse to Proposition 2.1.

Theorem 2.4. [LL03] The quotient ring $K_0(Var/\mathbb{C})/\mathbb{L}$ is naturally isomorphic to the free abelian group generated by stably birational equivalence classes of smooth projective connected varieties together with its natural ring structure.

In particular, if X and $Y_1, \ldots, Y_m$ are smooth projective connected varieties and

$$[X] \equiv \sum_{j=1}^{m} n_j [Y_j] \pmod{\mathbb{L}},$$

for some $n_j \in \mathbb{Z}$, then X is stably birational to one of the Y_j.

Remark. The proof relies on the Weak Factorization Theorem which is only known to be true in characteristic zero.

Note that the Grothendieck ring is not a domain [Poo02]. However there used to be a Cancellation conjecture which asserts that $\mathbb{L}$ is not a zero divisor in $K_0(Var/\mathbb{C})$. This has been proven false quite recently [Bor14].

3 Hilbert Scheme of Length Two Subschemes

In this section, we are going to understand the relation between Hilbert scheme of length 2 subschemes and 2nd symmetric powers.

Let X be a reduced quasi-projective variety of dimension d. Denote by $X^{[2]}$ the Hilbert scheme of length 2 subschemes of X. There is an open subvariety $X^{[2],0}$ inside X parametrizing reduced length 2 subschemes, i.e. 2 distinct points in X. Let $X^{[2]} \to X^{(2)}$ be the Hilbert-Chow morphism. It induces an isomorphism on the open subset $X^{[2],0} \simeq X^{(2)} - X$, where X is considered as the diagonal sitting inside $X^{(2)}$.

Next we are trying to understand the Hilbert-Chow morphism outside the open subsets.

Let $\mathcal{T}_X$ be the tangent sheaf of X, and $T_{x,X}$ be the fiber of $\mathcal{T}_X$ at $x \in X$. Due to upper semicontinuity of $\dim T_{x,X}$ ([Har77] Exercise II.5.8), we can define the following stratum of closed subvarieties of X:

$$Sing(X)_p = \{x \in X \,|\, \dim T_{x,X} = d + p\},\ p \geq 1.$$

Apparently, we can also define $Sing(X)_0$ in the same way, and it is the smooth locus of X, which is open inside X. Note that we give the reduced induced structure on $Sing(X)_p$. On each stratum $Sing(X)_p$, the tangent sheaf $\mathcal{T}_p := \mathcal{T}_X|_{Sing(X)_p}$ is of constant fiber dimension $d+p$. Since $Sing(X)_p$ is reduced, $\mathcal{T}_p$ is locally free of rank $d+p$ ([Har77] Exercise II.5.8 again).

Now let Z_p denote the non-reduced length 2 subschemes of X with support in $Sing(X)_p$. Then the Hilbert-Chow morphism restricting to Z_p is just $\mathbb{P}(\mathcal{T}_p) \to Sing(X)_p$, because Z_p parametrizes all tangent directions through a point in $Sing(X)_p$. By the fiber bundle formula (1.1), $[Z_p] = [\mathbb{P}^{d+p-1}][Sing(X)_p]$.

Lemma 3.1. If X is a hypersurface in a dimension $d+1$ smooth variety V, then

$$[X^{[2]}] = [X^{(2)}] + ([\mathbb{P}^{d-1}] - 1)[X] + \mathbb{L}^d [Sing(X)], \tag{3.1}$$

where $Sing(X)$ is the singular locus of X.

Proof. Because $\mathcal{T}_X \subset \mathcal{T}_V|_X$ and V is smooth, we have that $\dim T_{x,X} \leq \dim T_{x,V} = d+1$. Note that $Sing(X)$ is the union of all $Sing(X)_p$ for $p \geq 1$, so in this case, $Sing(X)_1 = Sing(X)$ and $Sing(X)_p = 0$ for $p > 1$.

Using the Hilbert-Chow morphism, we have

$$[X^{[2]}] - [Z_0] - [Z_1] = [X^{(2)}] - [X].$$

Therefore, by previous discussion, we can write

$$\begin{aligned}
[X^{[2]}] &= [X^{(2)}] - [X] + [Z_0] + [Z_1] \\
&= [X^{(2)}] - [X] + [\mathbb{P}^{d-1}][Sing(X)_0] + [\mathbb{P}^d][Sing(X)_1] \\
&= [X^{(2)}] - [X] + [\mathbb{P}^{d-1}][X] + ([\mathbb{P}^d] - [\mathbb{P}^{d-1}])[Sing(X)] \\
&= [X^{(2)}] + ([\mathbb{P}^{d-1}] - 1)[X] + \mathbb{L}^d[Sing(X)].
\end{aligned}$$

□

4 Fano Variety of Lines on a Cubic

In this section, we are going to look at an example to see how we can use the Grothendieck ring to study cubic hypersurfaces.

4.1 Definition

Let Y be a cubic hypersurface in $\mathbb{P}^{d+1} = \mathbb{P}(V)$, where V is a vector space of dimension $d+2$ and $\mathbb{P}(V)$ is considered as the space of lines through the origin. Let the defining equation of Y be

$$G \in \Gamma(\mathbb{P}^{d+1}, \mathcal{O}(3)) = Sym^3(V^\vee).$$

We consider the Grassmannian $Gr(2,V)$ to be the lines on $\mathbb{P}^{d+1}$ and let U to be the universal bundle on $Gr(2,V)$. Recall that U is a rank two subbundle of the trivial bundle $\mathcal{O}_{Gr(2,V)} \otimes V$. The section G induces a section $\widetilde{G} \in \Gamma(Gr(2,V), Sym^3(U^\vee))$ in the following way. For any point $L \in Gr(2,V)$, we can restrict $G \in Sym^3(V^\vee)$ to L to give an element $G|_L$ in $Sym^3(L^\vee)$, which is exactly the fiber over L of the bundle $Sym^3(U^\vee)$. Now the Fano scheme of lines on Y is defined to be the zero locus of the section $\widetilde{G}$, i.e.

$$Z(\widetilde{G}) \subset Gr(2,V).$$

$Z(\widetilde{G})$ is indeed the scheme parametrizing lines on Y. If L is a point in $Gr(2,V)$ which lies in the zero locus of $\widetilde{G}$, then $G|_L = \widetilde{G}(L) = 0$, and G vanishes along the line L exactly means that L lies in Y.

The Fano scheme may be nonreduced. There is an exercise in [Deb11] giving exmaples that are reducible and nonreduced. Here we will consider the reduced structure of it and denote the Fano variety by $F(Y) := Z(\widetilde{G})_{red}$. Note that the dimension of the Grassmannian $Gr(2,V)$ is $2d$, and the rank of the vector bundle $Sym^3(U^\vee)$ is $\binom{1+3}{3}=4$, so the expected dimension of $F(Y)$ is $2d-4$.

Here are some facts about $F(Y)$ listed without proving:

$F(Y)$ is connected if $d\geq 3$. $F(Y)$ may be singular or reducible. If Y is smooth, then $F(Y)$ is smooth of the expected dimension, and hence irreducible if $d\geq 3$. When Y is smooth, the canonical class of $F(Y)$ is $\mathcal{O}(4-d)$, where $\mathcal{O}(1)$ is given by the Plücker embedding

$$F(Y)\hookrightarrow Gr(2,V)\hookrightarrow \mathbb{P}(\bigwedge^2 V).$$

4.2 The $Y-F(Y)$ Relation

In this section, we are going to deduce a relation between a cubic hypersurface Y and its Fano variety of lines $F(Y)$ in the Grothendieck ring $K_0(Var/\mathbb{C})$.

Theorem 4.1. Let Y be a reduced cubic hypersurface in $\mathbb{P}^{d+1}$. We have the following relations in $K_0(Var/\mathbb{C})$:

$$[Y^{[2]}]=[\mathbb{P}^d][Y]+\mathbb{L}^2[F(Y)] \tag{4.1}$$

and

$$[Y^{(2)}]=(1+\mathbb{L}^d)[Y]+\mathbb{L}^2[F(Y)]-\mathbb{L}^d[Sing(Y)], \tag{4.2}$$

where $Sing(Y)$ is the singular locus of Y.

Proof. Consider the incidence variety

$$W:=\{(x,L)|x\in Y, L\subset\mathbb{P}^{d+1}, x\in L\}\subset Y\times Gr(2,V).$$

Note that W is exactly $\mathbb{P}(\mathcal{T}_{\mathbb{P}^{d+1}}|_Y)$, a $\mathbb{P}^d$-bundle over Y. Next we define a birational map ϕ between $Y^{[2]}$ and W as follows: For a point p in $Y^{[2]}$, it is either two distinct closed points on Y or a closed point with a tangent direction on Y. In any case, it determines a unique line L_p through p. Because Y is a cubic hypersurface, the intersection $L_p\cap Y$ is 3 points counting multiplicity for general choice of p. Therefore, there will be a third point x in $L_p\cap Y$ besides p, and we define $\phi(p)=(x,L_p)$. Note that the map ϕ is not defined exactly when the line L_p is contained in Y. Let Z be the closed subvariety of $Y^{[2]}$ consisting of those points p such that L_p is contained in Y. There is a projection $q:\ Z\to F(Y)$ sending p to $L_p\in F(Y)$.

We can also define the rational map ϕ^{-1} by $\phi^{-1}(x,L)=p$, where p is the length 2 subscheme of Y resulting from the intersection $L\cap Y$ subtracting x. ϕ^{-1} is also not defined exactly when L is contained in Y. Let Z' be the closed subvariety of W consisting of pairs (x,L) such that L is contained in Y. There is a projection $q':\ Z'\to F(Y)$ sending (x,L) to $L\in F(Y)$.

Now, outside Z and Z', ϕ gives an isomorphism between $Y^{[2]}$ and W, which means that $U:=Y^{[2]}-Z=U':=W-Z'$. Hence we have the following diagram:

$$\begin{tikzcd}
U \arrow[rr, "\simeq"] \arrow[d, hook] & & U' \arrow[d, hook] \\
Y^{[2]} \arrow[rr, dotted, "\phi"] & & W \\
Z \arrow[u, hook] \arrow[dr, "q"] & & Z'. \arrow[u, hook] \arrow[dl, "q'"] \\
& F(Y) &
\end{tikzcd}$$

In the diagram, the map $q' : Z' \to F(Y)$ is a $\mathbb{P}^1$-bundle because over each point $L \in F(Y)$ is the line $L \simeq \mathbb{P}^1$ inside Y. Similarly, the map $q : Z \to F(Y)$ is a $(\mathbb{P}^1)^{(2)}$-bundle over $F(Y)$, because over each point L is all length 2 subscheme of Y lying on L, i.e. all length 2 subscheme of $L \simeq \mathbb{P}^1$. Note that $(\mathbb{P}^1)^{(2)} \simeq \mathbb{P}^2$. Therefore, using the fiber bundle formula (1.1), we can rewrite W, Z and Z' as

$$\begin{aligned}
[W] &= [\mathbb{P}^d][Y] \\
[Z] &= [\mathbb{P}^2][F(Y)] \\
[Z'] &= [\mathbb{P}^1][F(Y)].
\end{aligned}$$

Now putting everything together we have

$$Y^{[2]} - [\mathbb{P}^2][F(Y)] = [\mathbb{P}^d][Y] - [\mathbb{P}^1][F(Y)].$$

Because $[\mathbb{P}^2] - [\mathbb{P}^1] = [\mathbb{A}^2] = \mathbb{L}^2$, we get

$$[Y^{[2]}] = [\mathbb{P}^d][Y] + \mathbb{L}^2[F(Y)].$$

(4.2) follows immediately from (4.1) by using the relation (3.1) in Lemma 3.1.

□

References

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *Journal of the American Mathematical Society*, 15(3):531–572, 2002.

[Bor14] Lev Borisov. Class of the affine line is a zero divisor in the grothendieck ring. *arXiv preprint arXiv:1412.6194*, 2014.

[Deb11] O. Debarre. *Higher-Dimensional Algebraic Geometry*. Universitext. Springer New York, 2011.

[GS14] Sergey Galkin and Evgeny Shinder. The fano variety of lines and rationality problem for a cubic hypersurface. *arXiv preprint arXiv:1405.5154*, 2014.

[Har77] R. Hartshorne. *Algebraic Geometry.* Encyclopaedia of mathematical sciences. Springer, 1977.

[LL03] Michael Larsen and Valery A Lunts. Motivic measures and stable birational geometry. *Mosc. Math. J*, 3(1):85–95, 2003.

[Poo02] Bjorn Poonen. The Grothendieck ring of varieties is not a domain. *Math. Res. Lett.*, 9(4):493–497, 2002.

[Wło03] Jarosław Włodarczyk. Toroidal varieties and the weak factorization theorem. *Inventiones mathematicae*, 154(2):223–331, 2003.