Algebraic v.s. Analytic Point of View

Ziwen Zhu

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In this talk, we will compare 3 different yet similar objects of interest in algebraic and complex geometry, namely algebraic variety, analytic variety and complex manifold. Then we will go through some deep and powerful theorems concerning the relation between these 3 objects. Finally, we will look at some examples and applications in order to have an idea about the benefit of switching among different viewpoints due to previous theorems. The base field will be $\mathbb{C}$ throughout the talk.

1 Varieties and manifolds

Intuitively, algebraic varieties are spaces that locally look like affine varieties. Indeed, we have the following definition:

Definition. A space X is an algebraic varieties if every $x \in X$ has a neighborhood U such that U is isomorphic to an affine variety.

We have Zariski topology on affine varieties, therefore the topology on an algebraic variety X is still in some sense "Zariski".

Now we want to define analogously analytic varieties. Simply put, we are replacing polynomials with holomorphic (analytic) functions. We first define analytic affine varieties.

Definition. Let $U \subseteq \mathbb{C}^n$ be the polydisc $\{|z_i| < 1 \mid i = 1, \dots, n\}$, An analytic affine variety X in U is the closed subset consisting of common zeros of $f_1, \dots, f_q$, where f_i's are holomorphic functions on U. We will denote $X = V(f_1, \dots, f_q)$.

Similar to the case of algebraic varieties, we can define the structure sheaf on an analytic affine variety X, i.e. the sheaf of holomorphic functions on X. Let $\mathcal{O}_X$ be the structure sheaf of X. Then $\mathcal{O}_X$ is the sheaf associated to the presheaf $\mathcal{O}_U/(f_1, \dots, f_q)$, where $\mathcal{O}_U$ is the sheaf of holomorphic functions on U, and $(f_1, \dots, f_q)$ is the ideal sheaf inside $\mathcal{O}_U$ determined by $f_1, \dots, f_q$. Intuitively, holomorphic functions on $X \subseteq U$ are locally the restriction to X of holomorphic functions on U, quotient those holomorphic functions that vanishes on X.

Now we can define analytic varieties as spaces that are locally analytic affine varieties. In many books, analytic varieties are also called analytic spaces.

Definition. A space X is an analytic variety if every $x \in X$ has a neighborhood U such that U is isomorphic to an analytic affine variety.

Finally, we give the definition of a complex manifold.

Definition. A complex manifold of dimension n is a real differentiable manifold of dimension $2n$ such that it admits an atlas $\{U_i, \varphi_i\}$, such that $\varphi_i : U_i \to \varphi_i(U_i)$ is a homeomorphism of U_i to open subsets of $\mathbb{C}^n$, and transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are holomorphic.

Let's look at a somewhat trivial example.

Example. The affine space $\mathbb{C}^n$ and the projective space $\mathbb{CP}^n$ are of course complex manifolds. Moreove, they are both algebraic varieties and analytic varieties as well because we can simply take them to be the vanishing locus of the zero function.

2 Relations between algebraic varieties, analytic varieties and complex manifolds

2.1 General Results

We have some quick and general results about the relations between all 3 types of objects.

First of all, by definition, we know that algebraic varieties are analytic varieties. Indeed, Suppose $X = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_q)$ is an affine variety. Because polynomials are holomorphic functions, X considered as the vanishing locus of $f_1, \ldots, f_q$ is an analytic space. Algebraic varieties result from gluing affine varieties, so they are analytic as well.

Next, we give a definition of regularity for analytic varieties (hence algebraic varieties), and then prove regular (nonsingular) varieties are complex manifolds. Thus a regular variety can in some sense be viewed as a "smooth" variety.

Definition. Let X be an analytic variety, a point $p \in X$ is called a regular point, if there is a neighborhood of $x \in U \subset \mathbb{C}^n$, and holomorphic functions $f_1, \ldots, f_q$ on U, such that $X \cap U = V(f_1, \ldots, f_q) \subset U$, and the matrix $\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{i \times j}$ has maximal rank (i.e. rank is q).

If every point in X is a regular point, then X is called regular (or nonsingular).

Locally, if X is the vanishing locus of $f_1, \ldots, f_q$ inside $U \subseteq \mathbb{C}^n$, we can define a holomorphic map $F : U \to \mathbb{C}^q$, given by $F(z_1, \ldots, z_n) = (f_1, \ldots, f_q)$. Then $X \cap U = F^{-1}(0)$. The condition that X is regular on U is the same as saying that 0 is a regular value of F, hence $X \cap U = F^{-1}(0)$ is a complex submanifold of U. We use holomorphic version of implicit function theorem here.

Example. Let's look at an example of hypersurface in $U \subseteq \mathbb{C}^n$, where U is the polydisc $\{|z_i| < 1 \mid i = 1, \ldots, n\}$. Suppose X is defined by a holomorphic function f on U, i.e. $X = V(f)$. If X is nonsingular at $p = (p_1, \ldots, p_n) \in X$, then by definition, all partials $\frac{\partial f}{\partial z_i}(p)$ are not zero. WLOG, we may assume $\frac{\partial f}{\partial z_n}(p) \neq 0$, then by IFT, we have neighborhoods of $(p_1, \ldots, p_{n-1})$ in $\mathbb{C}^{n-1}$ and p_n in $\mathbb{C}$, denoted by V_1 and V_2 respectively, and a holomorphic function $g : V_1 \to V_2$ such that $f(z_1, \ldots, z_{n-1}, g(z_1, \ldots, z_{n-1})) = 0$. Therefore, a neighborhood of p in X, namely $X \cap (V_1 \times V_2)$, is biholomorphic to V_1, an open subset in $\mathbb{C}^{n-1}$. The isomorphism is given by $(z_1, \ldots, z_{n-1}, z_n) \mapsto (z_1, \ldots, z_{n-1})$ and $(z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{n-1}, g(z_1, \ldots, z_{n-1}))$.

2.2 Chow's Theorem

We already know every algebraic variety is analytic. Then a natural question is whether every analytic variety is algebraic. This is generally false. There are 2 simple counterexamples due to the fact that analytic topology is finer than Zariski topology and there are more holomorphic functions than polynomials.

Example. Consider the polydisc in $\mathbb{C}^n$. It is analytic, but is not going to be defined by the zero locus of any set of polynomials because any polynomial that vanishes on polydisc will vanish on the whole $\mathbb{C}^n$.

This example results from an open subset in analytic topology that is not Zariski open.

Example. Consider the zero locus of $\sin z$ in $\mathbb{C}$, which consists of countably many points $\{k\pi, k \in \mathbb{Z}\}$. It is not the zero locus of a polynomial, or else it would be finite.

In this example, we find some holomorphic function which is not a polynomial.

However, Chow's theorem tells us that the answer to the question above is yes in projective spaces.

Theorem (Chow)**.** Let $X \subset \mathbb{CP}^n$ be an analytic subvariety, then X is algebraic.

This result is amazing because we start from spaces in the projective space that are locally defined by vanishing locus of holomorphic functions, but they turn out to be vanishing locus of polynomials globally.

Chow's theorem has an immediate corollary.

Corollary. Meromorphic functions on a projective variety is rational.

Proof. Given a meromorphic function f on a projective variety $X \subset \mathbb{CP}^k$, we can view it as a map from X to $\mathbb{CP}^1$. Let $\Gamma_f \subset X \times \mathbb{CP}^1$ be the closure of the graph of f. Then Γ_f is a subvariety of $X \times \mathbb{CP}^1$, hence a subvariety of $\mathbb{CP}^k \times \mathbb{CP}^1$, which embeds into $\mathbb{CP}^{k+1}$. Then by Chow's theorem, the graph Γ_f is algebraic. Let p and q be the 2 projections from $X \times \mathbb{CP}^1$ to X and $\mathbb{CP}^1$ respectively. Then p is an isomorphism from Γ_f to X, and f can be recovered as $f(x) = q(p^{-1}(x))$. Because p and q are algebraic, so is f. $\square$

There are some correspondence between some properties in algebraic and analytic varieties. For example, separatedness on the algebraic side is Hausdorff on the analytic side, etc.

2.3 GAGA

Serre's GAGA principle generalizes Chow's theorem, claiming that analytic coherent sheaves on a projective variety is actually algebraic, and that it doesn't matter whether we are calculating the cohomology of coherent sheaves in the analytic sense or in the algebraic sense.

We already know the definition of coherent sheaves on an algebraic variety. To state GAGA, we need to define as well analytic coherent sheaves.

First of all, let's look at a special case, the locally free sheaves (vector bundles) on a projective variety. Locally free sheaves are locally direct sums of the structure sheaf. For every projective algebraic variety X, we can view it as an analytic variety, denoted as X_h. And we associate to $\mathcal{O}_X$, the structure sheaf of X, the structure sheaf of X_h, denoted as $\mathcal{O}_{X_h}$. Then for every locally free sheaf $\mathcal{E}$ on X, we can associate an analytic locally free sheaf $\mathcal{E}_h$ in the following way:

For every local trivialization of $\mathcal{E}$, $\mathcal{E}|_U \simeq \mathcal{O}_U^r$, we can associate locally to $\mathcal{E}|_U$ the sheaf $\mathcal{O}_{U_h}^r$. Then we can glue them together for all U and get an analytic locally free sheaf $\mathcal{E}_h$.

Because coherent sheaf on an algebraic variety is defined in Hartshorne in an algebraic way, we need to find another way of definition in order to generalize the concept of coherent sheaf to analytic varieties. Now for a coherent sheaf $\mathcal{F}$ on X, locally on an affine open subset $U = \operatorname{Spec} A$, it is the sheaf associated to a finitely generated A-module M. A is Noetherian because it is finitely generated $\mathbb{C}$-algebra. Then we have the following exact sequence:

$$A^m \to A^n \to M \to 0.$$

Apply the $\sim$-functor, we have an exact sequence for sheaves:

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

That is every coherent sheaf is locally the cokernel of a map between free sheaves of finite rank. Then we can define in the same way the coherent sheaf on an analytic variety.

Definition. A sheaf $\mathcal{F}$ on an analytic variety X is coherent if for every $x \in X$, we have a neighborhood U such that we have the following exact sequence:

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

With coherent sheaf defined on analytic spaces, we can associate an analytic coherent sheaf to every algebraic coherent sheaf. Namely, for an algebraic coherent sheaf $\mathcal{F}$ on an algebraic variety X, we have locally $\mathcal{F}|_U = \operatorname{coker} \varphi$, where $\varphi : \mathcal{O}_U^m \to \mathcal{O}_U^n$. φ naturally induces a map $\varphi_h : \mathcal{O}_{U_h}^m \to \mathcal{O}_{U_h}^n$. Then we can associate locally the analytic sheaf $\operatorname{coker} \varphi_h$ and glue them together to get $\mathcal{F}_h$.

More strictly speaking, we can define the associated coherent sheaf in the following way: Let $f : X_h \to X$ be the identity map, which is continuous because analytic topology is finer than Zariski topology. Then we also have a natural map between structure sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X_h}$ by sending regular functions on X to itself on X_h considered as holomorphic functions. We have $f^*\mathcal{O}_U = \mathcal{O}_{U_h}$ for every open subset $U \subset X$. Then by applying f^* (which is always right exact) to the exact sequence:

$$\mathcal{O}_U^m \to \mathcal{O}_U^n \to \mathcal{F}|_U \to 0.$$

We know that we can define $\mathcal{F}_h = f^*\mathcal{F}$.

Remark. Actually here f^* is exact. Indeed, generally if we have an inclusion of open subset $i : U \to X$, i^* is right adjoint to $i_!$, which is exact.

f^* also defines natural maps between cohomology groups: $H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$. Indeed, if we have an injective resolution (in the category of sheaves of modules) $0 \to \mathcal{F} \to I^\bullet$, apply the exact functor f^*, we have an injective resolution $0 \to \mathcal{F}_h \to f^*I^\bullet$. Because topologically f is the identity map, we also have $\Gamma(X, \mathcal{G}) \to \Gamma(X_h, f^*\mathcal{G})$ for $\mathcal{G} = \mathcal{F}, I^i$ here. Therefore, we get a chain map between $\Gamma(X, I^\bullet)$ and $\Gamma(X_h, f^*I^\bullet)$, which induces the morphism of cohomology groups.

Now we can introduce Serre's GAGA principle. Simply put, it claims that every analytic coherent sheaf on a projective variety is algebraic. Furthermore, it doesn't matter how we calculate the cohomology of coherent sheaves. Formally, we have the following theorem:

Theorem (Serre)**.** Let X be a projective variety over $\mathbb{C}$, then there is an equivalence of categories from the category of coherent sheaves on X to the category of coherent analytic sheaves on X_h. Furthermore, for every coherent sheaf $\mathcal{F}$ on X, the natural maps

$$H^i(X, \mathcal{F}) \to H^i(X_h, \mathcal{F}_h)$$

are isomorphisms for all i.

Remark. The theorem is not true for sheaves that are not coherent. A simple counterexample is the constant sheaf $\mathbb{Z}$ on X.

Now Chow's theorem can be obtained directly from GAGA. Indeed, if $X \subset \mathbb{CP}^n$ is an analytic subvariety, we can associate to X an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{CP}^n}$ which is locally generated by the defining functions $f_1, \ldots, f_q$ of X. Thus $\mathcal{I}$ is an analytic coherent sheaf on X and by GAGA it is actually algebraic. Therefore, the subvariety defined by the ideal sheaf $\mathcal{I}$ is algebraic.

3 Applications

Chow's theorem and Serre's GAGA principle enable us to switch between algebraic and analytic point of views. On the algebraic side we can utilize a lot of results from commutative algebra to study algebraic varieties. Whereas on the analytic side, all the methods of complex analysis and differential geometry can be used. There are so-called "transcendental methods" referring to the process of using analysis to prove results for which no purely algebraic proofs are known.

Even for results that can be understand both algebraically and analytically, it is still helpful to switch between point of views. We list in this section some common ideas that both algebraic and analytic viewpoints come into play.

Example (Line bundles and the exponential sequence)**.** Let X be an analytic variety. We have an exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

where $\mathbb{Z}$ is the constant sheaf on X, and can be considered naturally a subsheaf of $\mathcal{O}_X$, and $\mathcal{O}_X^*$ is the sheaf of invertible (under multiplication) holomorphic functions. The map $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is given by sending f to $e^{2\pi i f}$. The sequence is indeed exact because $e^{2\pi z} = 1$ exactly when $z \in \mathbb{Z}$ and logarithm gives locally the inverse of the exponential map exp. This short exact sequence induces a long exact sequence of cohomology:

$$H^{i-1}(X, \mathcal{O}_X^*) \to H^i(X, \mathbb{Z}) \to H^i(X, \mathcal{O}_X) \to H^i(X, \mathcal{O}_X^*) \to H^{i+1}(X, \mathbb{Z}).$$

Now if we consider the case that X_h is analytic projective, whose underlying algebraic variety is X. Then because $H^0(X_h, \mathcal{O}_{X_h}) = H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X_h, \mathcal{O}_{X_h}^*) = H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$, for the H^0 part of the long exact sequence, we have

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0.$$

Therefore, starting from H^1, we have

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}_{X_h}^*) \to H^2(X_h, \mathbb{Z}) \to \dots.$$

First we notice that there is an isomorphism $H^1(X_h, \mathcal{O}_{X_h}^*) = \operatorname{Pic} X_h$. Indeed, by first viewing $H^1(X_h, \mathcal{O}_{X_h}^*)$ as Čech cohomology on an open cover $\{U_i\}$ of X_h, an element in $H^1(X_h, \mathcal{O}_{X_h}^*)$ is a collection of cocyles $\{(U_{ij}, g_{ij})\}_{i,j}$. Then in the Čech complex, $\{(U_{ij}, g_{ij})\}_{i,j}$ is closed means that all g_{ij} satisfies cocycle condition $g_{ik} = g_{ij}g_{jk}$. And it is exact if we have $g_i \in \mathcal{O}_{U_i}^*$ such that $g_{ij} = g_j g_i^{-1}$. Therefore, closedness tells us g_{ij} defines a line bundle on X_h, whereas exactness tells us g_{ij} defines the trivial line bundle $\mathcal{O}_{X_h}$. Therefore we have an injective map from $H^1(X_h, \mathcal{O}_{X_h}^*)$ to $\operatorname{Pic} X_h$. Moreover, since every line bundle can be determined by transition functions g_{ij} which can be viewed as cocycles in $H^1(X_h, \mathcal{O}_{X_h}^*)$, we know $H^1(X_h, \mathcal{O}_{X_h}^*) = \operatorname{Pic} X_h$.

By GAGA, we then have $\operatorname{Pic} X_h = \operatorname{Pic} X$, i.e. analytic line bundles over X_h are the same as algebraic line bundles over X. We also have $H^1(X_h, \mathcal{O}_{X_h}) = H^1(X, \mathcal{O}_X)$. So the sequence we have now becomes

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic} X \to H^2(X_h, \mathbb{Z}) \to \dots.$$

Now we can easily define the fisrt chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L} \in \operatorname{Pic} X$ to be its image under the map $\operatorname{Pic} X \to H^2(X_h, \mathbb{Z})$. Also if we denote by $\operatorname{Pic}^\circ X$ the line bundle that are numerically trivial (in most cases we may assume this is the same as saying $c_1(\mathcal{L}) = 0$), we can deduce from the long exact sequence that $\operatorname{Pic}^\circ X \simeq H^1(X, \mathcal{O}_X)/H^1(X_h, \mathbb{Z})$. Indeed,

$$\begin{aligned} \operatorname{Pic}^\circ X &= \ker(\operatorname{Pic} X \to H^2(X_h, \mathbb{Z})) \\ &= \operatorname{coker}(H^1(X_h, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)) \\ &= H^1(X, \mathcal{O}_X)/H^1(X_h, \mathbb{Z}). \end{aligned}$$

When X is a nonsingular curve of genus g, or equivalently, X_h is a Riemann surface of genus g, we can have a more explicit expression for $\operatorname{Pic}^\circ X$. Now $H^1(X_h, \mathbb{Z}) = \mathbb{Z}^{2g}$ because topologically X_h is a sphere with g handles. On the other hand, by Serre duality, we have $H^1(X, \mathcal{O}_X) \simeq H^0(X, \omega_X)' \simeq \mathbb{C}^g$. So

$$\operatorname{Pic}^\circ X \simeq \mathbb{C}^g/\mathbb{Z}^{2g},$$

which is a complex torus. This is the Jacobian variety of the curve X.

Another advantage of the analytic viewpoint is that we can define metrics on a complex manifold and the vector bundles on it, Which will relate many concepts to solving a specifi PDE and hence allow us to use analysis approaches.

Example (Hodge decomposition)**.** First of all, in classical Hodge theory, on a compact Kähler manifold X, we have Hodge decomposition

$$H^m(X, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(X).$$

The decomposition briefly speaking is given in the following way: by viewing $H^m(X, \mathbb{C})$ as De Rham cohomology, we can find a unique harmonic representative for every element in $H^m(X, \mathbb{C})$, and then we can decompose it into harmonic (p, q)-forms with $p+q = m$. The notion of harmonic forms is defined by a Laplace operator Δ associated to the Kähler metric on X, therefore in general, we don't have Hodge decompostion for algebraic varieties.

However, for nonsingular algebraic projective varieties, we actually have Hodge decomposition. Indeed, for projective spaces $\mathbb{CP}^n$, there is a standard Fubini-Study metric on it which is Kähler. Therefore, for every projective variety X, we have the induced Kähler metric pullbacked from some $\mathbb{CP}^n$, making X into a Kähler manifold. So there is Hodge decomposition on X.

There are some theorem later in Hodge theory claiming that every compact algebraic variety (not necessarily projective) admits a Hodge decomposition though it might not be Kähler.

Example (Ample line bundles)**.** The ampleness of a line bundle $\mathcal{L}$ on an algebraic variety X can be defined in the following way: First of all, $\mathcal{L}$ is very ample if there exists a closed immersion $X \subset \mathbb{CP}^N$ such that

$$\mathcal{L} = \mathcal{O}_{\mathbb{CP}^N}(1)|_X.$$

Then a line bundle $\mathcal{L}$ is called ample if $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$.

On the other hand, there is a similar notion of positivity of a line bundle on analytic variety. Given a Hermitian metric h on a Line bundle $\mathcal{L}$ on X, it determines a curvature form $\Theta(\mathcal{L}, h)$ which is a (1,1)-form (Moreover, $\frac{i}{2\pi}\Theta(\mathcal{L}, h)$ is actually a representative of $c_1(\mathcal{L})$). Then the line bundle $\mathcal{L}$ is called positive if it carries a Hermitian metric h such that $\frac{i}{2\pi}\Theta(\mathcal{L}, h)$ is a Kähler form (i.e. the (1,1)-form associated to a Kähler metric).

We have the following Kodaira embedding theorem claiming that positive line bundle is the same as ample line bundle.

Theorem (Kodaira)**.** Let X be a compact Kähler manifold, and $\mathcal{L}$ a holomorphic line bunlde on X. Then $\mathcal{L}$ is positive if and only if there is a holomorphic embedding

$$\phi : X \hookrightarrow \mathbb{CP}^N$$

of X into some projective space such that $\phi^*\mathcal{O}_{\mathbb{CP}^N}(1) = \mathcal{L}^m$ for some $m > 0$.

By GAGA, we know from Kodaira embedding theorem that if we have a positive line bundle $\mathcal{L}$ on a compact Kähler manifold X, then both X and $\mathcal{L}$ are algebraic and $\mathcal{L}$ is ample.

Fianlly, let's look at an example about how to resolve by an algebraic approach the problem that Zariski topology is too coarse. In some sense, the method is actually inspired from the analytic point of view.

Example (Completion of a local ring)**.** Let X and Y be two algebraic varieties. Suppose there are points $p \in X$ and $q \in Y$ such that the local rings of germs of regular functions $\mathcal{O}_{p,X}$ and $\mathcal{O}_{q,Y}$ are isomorphic as $\mathbb{C}$-algebras. Then there are open sets $p \in U \subset X$ and $q \in V \subset Y$ and isomorphism of U to V which sends p to q. This in particular shows that X and Y are birationally equivalent. Therefore, on an algebraic variety, local ring at a specific point pretty much tells us almost everything about the variety itself, and we are actually not looking at something really local when studying the local ring.

However, we can introduce $\hat{\mathcal{O}}_{p,X}$, the completion of the local ring $\mathcal{O}_{p,X}$, which intuitively is replacing germs of regular functions with germs of power series (not necessarily convergence, contrary to the case of germs of holomorphic functions). Then we can see from the following fact that $\hat{\mathcal{O}}_{p,X}$ indeed forgets more global properties and reflects information that is much more local.

Observation. If $p \in X$ and $q \in Y$ are nonsingular points on varieties of the same dimension, then $\hat{\mathcal{O}}_{p,X}$ is isomorphic to $\hat{\mathcal{O}}_{q,Y}$.

This observation is the algebraic analogue of the fact that any two manifolds of the same dimension n are loccally isomorphic (in fact to a polydisc $U \subset \mathbb{C}^n$).