The Power of Calculus - Unexpected and Deep Ideas from Calculus

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1 Old Ideas

I'll begin this talk by discussing some old ideas, many of them ideas with which you are probably already familiar. However, it's important, interesting, and fun to review these ideas, as they will be the basis for much of our later discussion.

These are some basic and famous results from classical mathematics. They include some of the earliest results in geometry and number theory.

1.1 Prime Numbers

A prime number, p, is a positive integer that is not divisible by any smaller positive integer except 1. A composite number is any number that is not a prime number. The product of prime numbers equal to a given positive integer n is called the prime decomposition of n.

1.1.1 Unique Decomposition

The prime decomposition of a number n is unique.

Proof

Suppose n had two different prime decompositions:

$$n = p_{\alpha_1}^{k_{\alpha_1}} p_{\alpha_2}^{k_{\alpha_2}} \cdots p_{\alpha_q}^{k_{\alpha_q}}$$

and

$$n = p_{\beta_1}^{k_{\beta_1}} p_{\beta_2}^{k_{\beta_2}} \cdots p_{\beta_r}^{k_{\beta_r}}$$

If we equate these two numbers and divide by all their common primes we get:

$$p_{\alpha_1}^{k'_{\alpha_1}} p_{\alpha_2}^{k'_{\alpha_2}} \cdots p_{\alpha_q}^{k'_{\alpha_q}} = p_{\beta_1}^{k'_{\beta_1}} p_{\beta_2}^{k'_{\beta_2}} \cdots p_{\beta_r}^{k'_{\beta_r}}$$

Now, if we divide both sides by p_{α_1} we see that if $k'_{\alpha_1} \neq 0$ then p_{α_1} divides the left hand side, and so much divide the right hand side. However, as every term on the right hand side is prime this implies that p_{α_1} divides a prime number on the right hand side, and this cannot be as we assumed we've already divided all the common primes. So, $k_{\alpha_1} = 0$, and similar logic can be applied to every exponent, and so both sides of the above equation must be 1. Therefore, all the prime numbers between the two representations of n are the same, and therefore the prime decompositions are the same. So, the prime decomposition is unique. Q.E.D.

We note also that any given composition of prime numbers gives us a positive integer. So, the compositions of prime numbers are in bijective correspondence with the positive integers.

1.1.2 Infinitude of Primes

There are an infinite number of primes. This can be seen in a classical proof by contradiction from Euclid.

Proof

Suppose there are a finite number of primes $p_1, p_2, \ldots, p_N$. Then if we take the product of all these primes and add 1 we get a number:

$$Q = p_1 p_2 \cdots p_N + 1$$

We note that Q cannot be prime as it must be larger than all prime numbers. However, it also cannot be divisible by any prime, because all primes would divide the first term on the right, but none would divide the second. As all numbers are either prime or not we have a contradiction, and therefore the number of primes is infinite. Q.E.D.

Note we will construct a very different proof of this fact at the conclusion of this lecture.

1.2 Zeno's Paradox of the Arrow

Suppose you fire an arrow at a target. It has to go half the distance between yourself and the target, and then half the remaining distance, and then half the remaining distance, and then .... As there are an infinite number of terms in this sequence, and yet only a finite distance the arrow travels, how is this possible? How can you add up an infinite number of positive numbers, and still get a finite number? For that matter, how the heck do you make sense of adding up an infinite set of finite numbers anyway?

Well, as you all probably know Zeno's paradox can be essentially resolved through the concept of an infinite series. In this case, a particular type of series called a geometric series.

We define an infinite sum in terms of partial sums and limits. In the case of the geometric series, we can find a closed form solution for the partial sum, and take the limit as the partial sum goes to infinity to get the infinite sum.

For the geometric series we have a series of the form:

$$\sum_{n=1}^{\infty} x^n$$

where the partial sum function can be defined as:

$$S(N) = \sum_{n=1}^{N} x^n$$

Now, using an old trick, if we multiply the partial sum function by x and then subtract it from itself we get:

$$xS(N) = \sum_{n=2}^{N+1} x^n$$

and

$$xS(N) - S(N) = x^{N+1} - 1$$

where all the inner terms cancel in pairs. This is known as telescoping. Solving this for $S(N)$ we get:

$$(x-1)S(N) = x^{N+1} - 1$$
$$\rightarrow \frac{1 - x^{N+1}}{1-x}$$

Now, if we take the limit here as $N \rightarrow \infty$ we see that $S(N)$ is convergent if $|x| < 1$ and divergent if $|x| \geq 1$. If $|x| < 1$ we get:

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

2 New Idea

We can define a different infinite series as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This is the *Riemann zeta function*. Note that for $s > 1$ the integral convergence test tells us that the series is convergent:

$$\int_1^{\infty} \frac{1}{x^s} = \frac{x^{1-s}}{1-s}|_1^{\infty} = \frac{1}{s-1}$$

while for $s = 1$ the integral test tells us the series is divergent:

$$\int_1^{\infty} \frac{1}{x} = \ln x|_1^{\infty} = \infty.$$

This divergent series is know as the *harmonic series*. Now, if we note that for $s > 1$ and any prime number p, as all prime numbers are greater than 1, our formula for the geometric series tells is:

$$\sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \frac{1}{\left(1 - \frac{1}{p^s}\right)} = \left(1 - \frac{1}{p^s}\right)^{-1}$$

There is less here than meets the eye. This is just noting a specific case of our more general formula for the geometric series.

OK, well, so what. Everything I've done so far is stuff you're probably already familiar with. Well, here's the critical, amazing, out of left field, spark of the divine, flash of brilliance insight that makes this worth while.

We note that in the Riemann zeta function we're summing over all numbers. So, we can rewrite this as a sum over all possible prime decompositions of numbers:

$$\zeta(s) = \sum_{p=prime} \sum_{k=0}^{\infty} \frac{1}{p_1^{ks} p_2^{ks} \ldots}$$

Now, this can be rewritten as:

$$\sum_{p=prime} \sum_{k=0}^{\infty} \frac{1}{p_1^{ks} p_2^{ks} \ldots} = \sum_{k=0}^{\infty} \frac{1}{p_1^{ks}} \sum_{p \neq p_1} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \left(1 - \frac{1}{p_1^s}\right) \sum_{p \neq p_1} \sum_{k=0}^{\infty} \frac{1}{p^{ks}}$$

If we do this for every prime we see that we can write the Riemann zeta function as:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is over all prime numbers. Wow!

As noted earlier, the zeta function diverges for $s = 1$. However, based on our above equality (for which we did not use the convergence of $\zeta(s)$ in the proof, and which is valid for other values of s as well) we see that if the number of prime numbers were finite, then $\zeta(1)$ would have to be finite. So, as $\zeta(1)$ is infinite, there must be an infinite number of primes. Pretty cool, huh?

3 More Old Ideas

Here we will again discuss how some more modern ideas, namely recurrence relations, allow us to solve some old problems, and reveal some startling connections with very old ideas.

3.1 The Golden Ratio

Take a line l, and divide it into two parts a and b, such that the ratio of l to b is the same as the ratio of b to a. What is the length of l as a fraction of b?

Well, this is just a simple algebra problem:

$$l = a + b, \text{ so } a = l - b.$$

Now, we require b satisfy the relation:

$$\frac{l}{b} = \frac{b}{a}$$
$$\rightarrow \frac{l}{b} = \frac{b}{l-b}$$
$$\rightarrow l^2 - lb = b^2$$

and so

$$\rightarrow \left(\frac{l}{b}\right)^2 - \frac{b}{l} - 1 = 0$$

which if we solve using the quadratic equations we get:

$$\frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{1+\sqrt{5}}{2} = \phi$$

where we have taken the positive term as we require our answer to be positive.

The number ϕ so defined is the famous Golden ratio that was so beloved by the Greeks.

3.2 The Fibonacci Sequence

Moving ahead about 1500 years we encounter the Fibonacci sequence, which was introduced to the West by Leonardo of Pisa as a hypothetical model of rabbit populations. The sequence goes like:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

and it is defined recursively as:

$$x_{n+2} = x_{n+1} + x_n$$

with initial conditions:

$$x_0 = 0$$
$$x_1 = 1$$

4 Cool Idea

Now, this recurrence relation along with these initial conditions completely determines the sequence, and in theory if we were asked to figure out any term, say $x_{5,876,259}$, we could do so just by starting with the first two terms and building up. However, a natural question to ask is whether there is any closed form solution to x_n. In other words, a formula that would just tell us the value without requiring us to compute all previous values.

Well, we can do this by first taking a "guess" at the form our answer will take. We'll guess that our answer will be of the form:

$$x_n = r^n$$

Now, if this is the case then our recurrence relation tells us:

$$r^{n+2} = r^{n+1} + r^n$$

or, dividing through by r^n

$$r^2 = r + 1$$

which has solutions:

$$r = \frac{1 \pm \sqrt{5}}{2}$$

Look familiar? We note that either of these solutions would satisfy the recurrence relation. In fact, any linear combination of these two solutions would satisfy the recurrence relation. So, how to we pick which one to use? Well, in a situation tantalizing similar to the situation with second order differential equations, we use our two initial conditions.

The general form of our solution is:

$$x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

So, our initial values tell us:

$$A + B = 0$$
$$\text{and}$$
$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

which give us:

$$A = \frac{1}{\sqrt{5}} \text{ and } B = -\frac{1}{\sqrt{5}}$$

So, our closed form solution will be:

$$x_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Now, an interesting question here is what happens if we look at the ratio of consecutive terms. That is, the sequence:

$$y_n = \frac{x_{n+1}}{x_n} \text{ for } n > 0$$

Well, the limit of these terms as $n \to \infty$ is:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$
$$= \lim_{n\to\infty} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2((1+\sqrt{5})^n - (1-\sqrt{5})^n)} = \lim_{n\to\infty} \frac{(1+\sqrt{5}) - (1-\sqrt{5})\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{2(1+\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n)}$$
$$= \frac{1+\sqrt{5}}{2} = \phi$$

So, ϕ shows up where you'd least expect it!