Knots, Links, and Invariants - A Basic Overview of Knot Theory

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Today I'm going to talk about the basics of knot theory: what we mean when we talk about a knot or its generalization, a link, what questions knot theoriests try to answer, and what techniques they've developed for answering these questions. We'll introduce one of the major concepts in knot theory, the idea of an invariant, and introduce two invariants that are not obvious. One is a pretty simple invariant, and the other is a very powerful invariant that's only been discovered in the last 30 years.

1 The Basics

1.1 Definiton

So, what is a knot? In mathematical terms a knot is a simple closed curve in 3-space with no self intersections and no boundary. Usually when we're dealing with a knot, for practical reasons if for no other, we deal with its projection onto 2-space. So, for example, the simplest knot is just a simple loop. This is called the unknot, and its simplest projection is:

Now, we consider two knots to be the same if we can stretch and move one of them around in 3 space in such a way that we don't have to cut it, but still at the end of our moves it then looks like the other knot. So, depending on how we stretch it and move it around, one knot can have many, many different projections (all knots, in fact, have an infinite number of possible projections). For example, this knot:

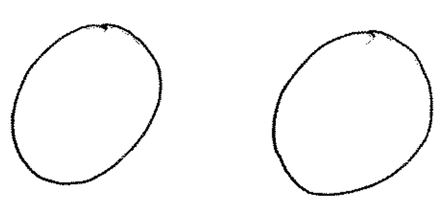
is also the unknot. So, a major question in knot theory is, given two projections that are not obviously the same, how do we tell that they're not the same? Probably the second simplest knot is the standard trefoil knot:

but how do we know that this isn't just another projection of the unknot? (In this case it's rather intuitively obvious, but intuition is not a mathematical proof, and in many harder cases our intuition may be wrong.)

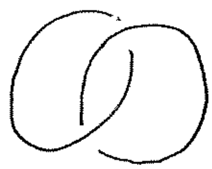
We will address this question and other questions like it when we discuss the concept of invariants.

1.2 Links

We will also deal with links, which is just a generalization of the concept of a knot. A link is just a set of multiple knots which are potentially tied together. So, for example, we could have the 2-unlink:



Or perhaps the simplest nontrivial link made from two distinct curves, the Hopf link:

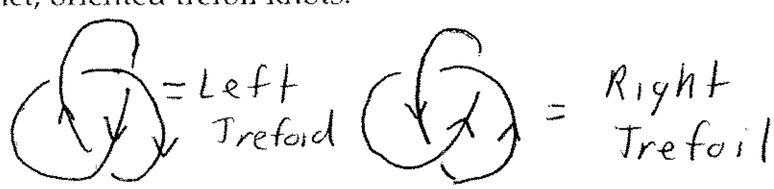


Or a link with 3 curves known as the Borromean rings:

the name Borromean rings comes from the coat of arms of the Borromeas, a prominant Italian family during the Renaissance. We note that the Borromean rings are an example of a type of link called a Brunnian link, where if you remove one of the curves, all the other curves can separate.

1.3 Orientations

Now, for any knot or any link we can provide an orientation. You just set a consistent direction around every curve. So, for example, you can have two distinct, oriented trefoil knots:

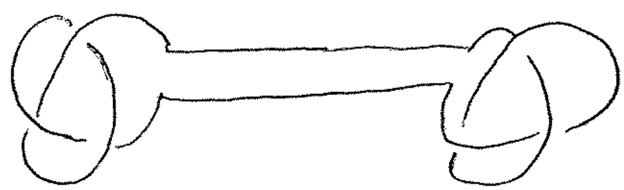


Or two distinct oriented links:

2 Prime and Composite Knots

2.1 Composition

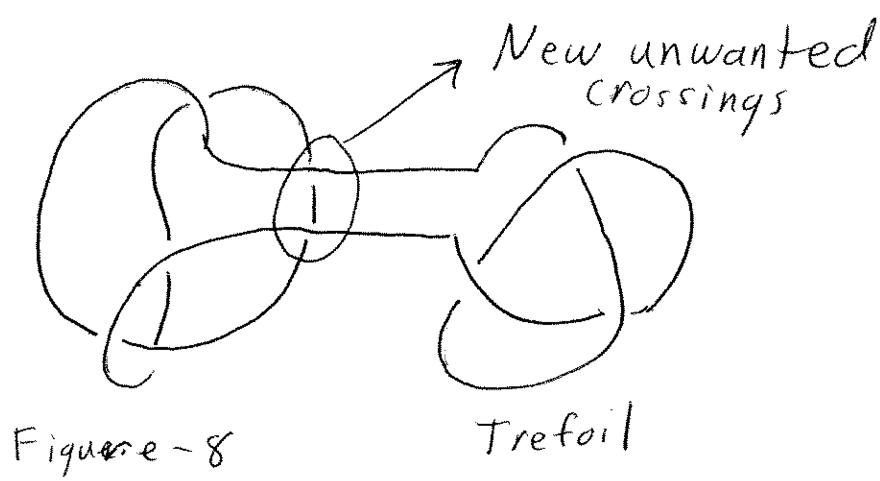
Note that we can define a concept of composition, multiplication if you will, of two knots. You just remove a small segment from the curve of each knot, and connect the two knots along this removed segment:



so, this knot above is the composition of two standard trefoil knots.

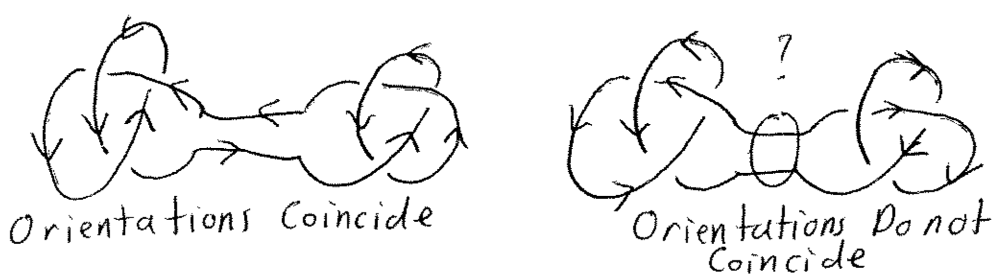
2.2 Difficulties

Now, one important thing here is that you have to be careful where you take our the two segment, because depending on where you take our the two segments you might get two different knots. We restrict our segments in such a way that we require that they be taken from the "outside" of the knot. More specifically, we require that the composition of two knots does not introduce any new crossings that were not already present in the original knot. So, for example, the "composition" below would not be allowed:



2.3 Orientations

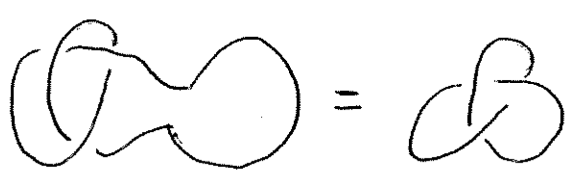
Now, even with this restriction, you still might get two compositions depending on where you take our the segments. It can be proven that when taking the composition of two knots there can be at most two possibilities. These possibilities correspond to if you give both knots an orientation, and then compose them, whether or not these orientations form a consistent orientation on the composition knot. So, in general there are two possible composition knots.



However, there are some oriented knots that are equivalent to themselves with orientation reversed. These knots are called amphicheiral. If one of the knots in a "product" is amphicheiral, then there's only one possible product knot.

2.4 Composite Knots

First we note that if we take any knot and compose it with the unknot, we just get the knot back:



So, the unknot serves as a "unit" when dealing with these compositions. Now, if a knot can be written as the composition of two knots, where neither is the unknot, then we consider the knot to be a composite knot, and the two knots that multiply together to form it "factor knots". If a knot is not a composite knot then we cann it, can you guess, a prime knot. The trefoil is one example of a prime knot. Now, something that we won't get into right now is that you can make an analogy with this situation and the situation we see in number theory with prime and composite numbers. In fact, you can prove a version of unique factorization for composite knots that's analogous to unique factorization in the integers.

3 Equivalence

3.1 Basic Idea

As mentioned earlier, two knots (or more generally two links) are considered equivalent if we can start with one and move the curves around in such 3-space in such a way that we can stretch them or shrink them and bend them any way we want so long as we don't pass a curve through itself or cut it and end up with the other one.

So, for example, these knots are equivalent:

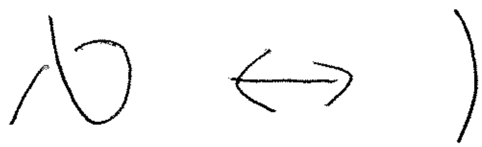
while these are not:

However, while it's possible to show that the first set of knots are equivalent, how do we show that the second set isn't? It isn't obvious that they're equivalent, and if you play around with them for a while you'd probably guess they're not equivalent, but that's not a mathematical proof. Formal methods for answering this problem will be the major focus of most of the rest of this talk, but first we need to introduce a very useful theorem in knot theory. This theorem involves something called Reidemeister moves.

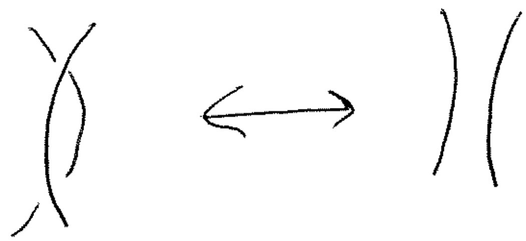
3.2 Reidemeister Moves

There are 3 types of Reidemeister moves:

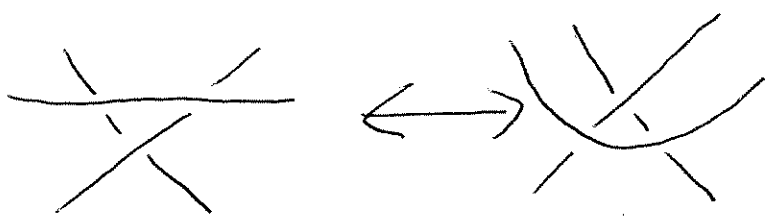
Type 1:



Type 2:



Type 3:



Now, the important idea is that in 1926 Kurt Reidemeister proved that these moves are all you need! In other words, if you have two projections of the same knot, you can always turn one projection into the other projection through a sequence of Reidemeister moves. Now, the proof of this is not particularly difficult, but from what I understand (I've never read it) it's rather long and involved and boring. So, we won't go into why it's true here, and we'll just trust that it is in fact true.

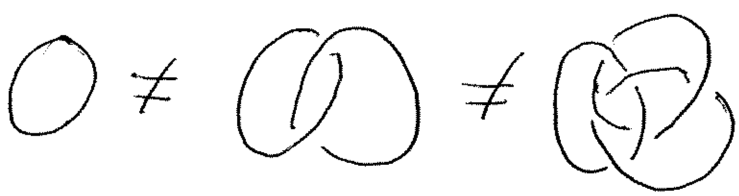
Now, it turns out that this fact is a wonderful workhorse, and will allow us to figure out a number of things called "invariants", which will be the major properties that we'll use to tell knots apart.

4 Invariants

When we assign an invariant to a knot or a link we mean that there is a definite, unambiguous way of assigning to any knot an element of some set (it could be a number or, as we'll see later, a polynomial) such that if two knots are assigned a different element from that set, then we know the two knots are distinct. Now, it may be that two knots have the same value for a given invariant, but the two knots are still distinct. An invariant doesn't necessarily tell you when two knots are the same, but it can tell you when two knots are different.

4.1 The Number of Curves

One very simple invariant when you're talking about links is just the number of curves there are in your link. A link made with, for example, 3 curves is definitely going to be distinct from a link made with 2 curves. So, using some knots we saw earlier, the unknot, the Hopf link, and the Borromean rings are all distinct links:

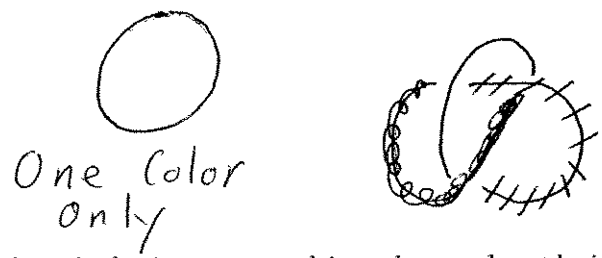


However, obviously, all knots are not the same, even though they all consist of just one curve. So, just because two links have the same number of curves doesn't mean they're the same.

4.2 Tricolorability

This is our first example of what might be called a nonobvious or nontrivial invariant. We consider a knot to be tricolorable if for a given projection of the knot we can color the different sections of its curves such that we use the same color in between crossings, and at any given crossing we either have three distinct colors coming together, or the same color coming together. We also require for a knot to be tricolorable that we use at least two colors in the tricoloration.

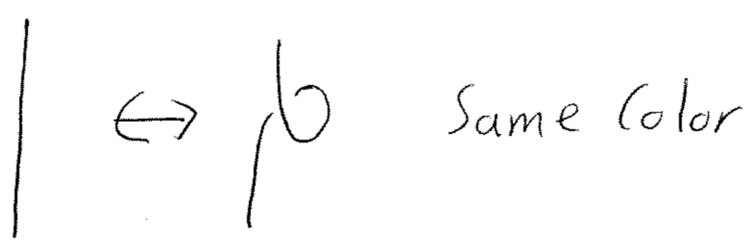
So, for example, the unknot is not tricolorable, while the trefoil is:



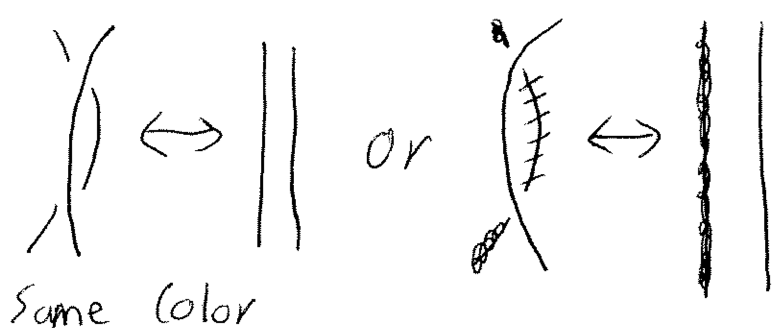
Now, here's the important thing about a knot being tricolorable or knot. It's an invariant. That is, if a knot is tricolorable in one of its projections, it's tricolorable in all of it's projections. This give us a mathematically rigorous proof that the unknot and the trefoil knot are in fact distinct knots.

How do we prove this? Well, we just see that tricolorability can be maintained under any of the three Reidemeister moves. The proof for the third takes a number of cases and we won't get into it here (trust me it works) but the proof for the first and second Reidemesiter moves is very quick to see:

Move 1:



Move 2:



4.3 The Jones Polynomial

I'll end today by talking about the Jones polynomial. It's an invariant that's easy to calculate, and it's far more powerful than the number of links or tricolorability. (We note that tricolorability, for example, only sorts knots into two categories, and there are more than two types of knots!)

Now, the Jones polynomial assigns to each knot a Laurent polynomial in an unambiguous and always calculable manner. Now, this calculation can take a long time, but it's always possible and always takes a finite amount of time. Also, for the same knot it will always yield the same result, regardless of the projection. So, how does this work? Let's find out.

4.4 Constructing the Jones Polynomial

We begin by considering projections of unoriented links. For a given projection L we define a Laurent polynomial using the 3 rules:

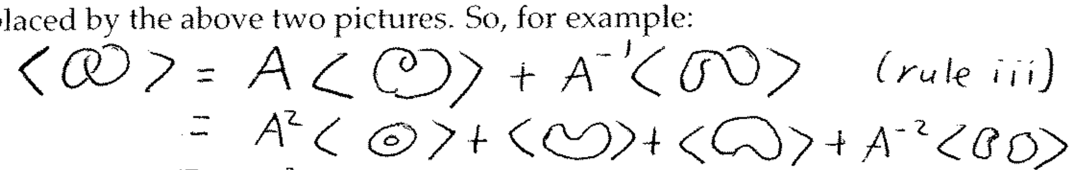
i) $\langle O \rangle = 1$

ii) $\langle L \cup O \rangle = -(A^{-2}+A^{2})\langle L \rangle$

iii) $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$

where here $L \cup 0$ is the union of a knot L with the unknot, where there are no crossings:

while $\langle \asymp \rangle$ and $\langle)(\rangle$ represent the same knot as $\langle \times \rangle$ with just the crossing replaced by the above two pictures. So, for example:



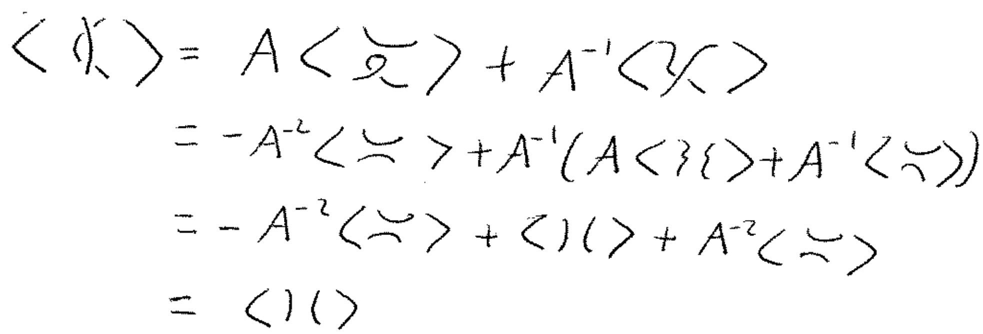
$= -A^2(A^{-2}+A^2) + 1 + 1 + A^{-2}(A^{-2}+A^2)$

$= -(A^4 - A^{-4})$

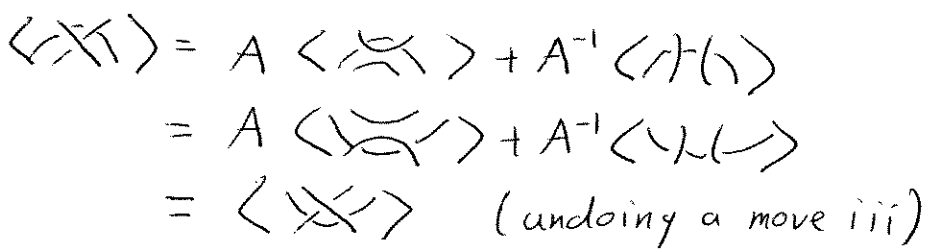
Now, repeated application of the third rule reduces the number of crossings until there are no crossings at all, and then the first two rules do the cleanup. So, any knot can in this way be reduced. (We note that it doesn't matter the order in which the crossings are handled.)

OK, so how do we prove that it's an invariant? We look at how the polynomial changes under the Reidemeister moves. So:

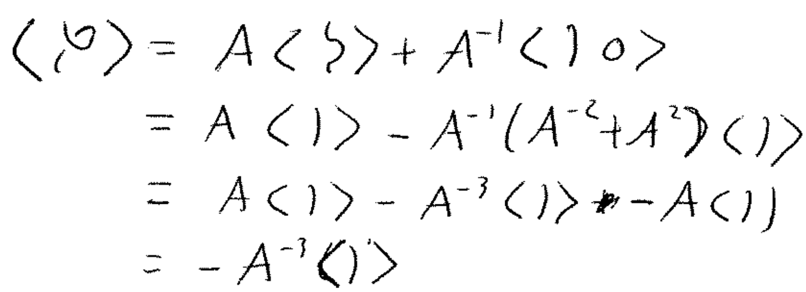
Move 2:



Move 3:

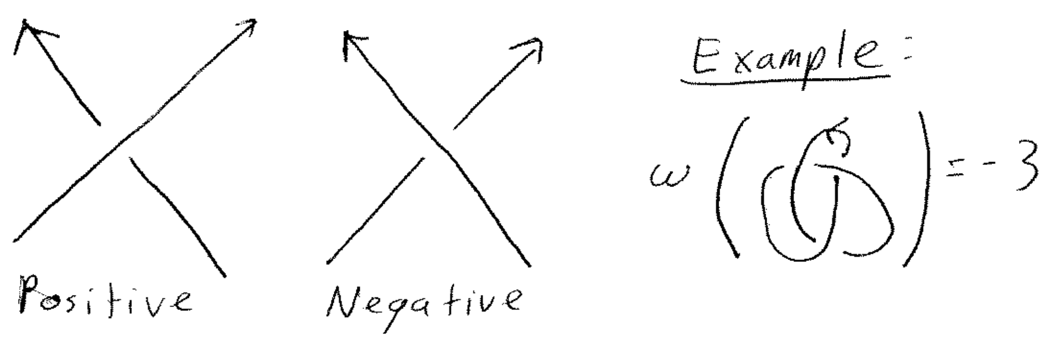


Move 1:



Uh Oh! What happens to move 1? It doesn't work. So, does this mean the Jones polynomial is wrong? No. Fortunately, this is salvageable. Here's how.

If we provide our knot L with an orientation we can define $w(L)$, the write of L, as the algebraic sum of the crossings of L, counting $+1$ for a positive crossing and -1 for a negative crossing. The first Reidemeister move subtracts on from the write, so it's not invariant under the first move, but it will be invariant under the second and third Reidemeister moves. Note that positive and negative are defined as:



So, finally, the polynomial:

$$X(L) = (-A)^{3w(L)} < L >$$

will be invariant under all Reidemeister moves. So, is this our Jones polynomial? Not quite. If we substitute $A = t^{-\frac{1}{4}}$, which provides the polyomial with some computationally useful algebraic properties, we get the Jones polyomial.

As mentioned earlier, while the Jones polynomial may take a while to calculate, it can be a very powerful invariant that can be used to distinguish a number of knots.

Now, there are, as you might imagine, many more invariants and interesting properties of knots that are worth studying. If you're interested, I'd recommend starting with "The Knot Book" by Colin C. Adams. It's a great book that covers all of this and more at a level that is definitely accessible to the standard undergraduate math major. Thanks!