Symmetric Tropical Matrices

Dylan Zwick

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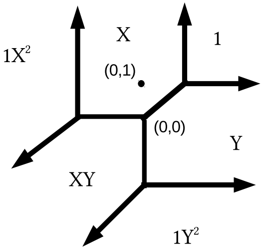

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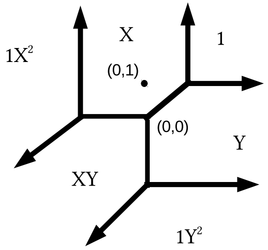

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is not singular.

Outline

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Also, we'll be working over $\mathbb{R}$, and not the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. In particular, this will mean later on that all matrices have rank at least one.

Tropical Polynomials and Tropical Hypersurfaces

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  (tropical addition and multiplication), and represents a piecewise linear convex function $F : \mathbb{R}^m \to \mathbb{R}$.
- The *tropical hypersurface* $\mathbf{V}(F)$ defined by a tropical polynomial F is the set of all points $P \in \mathbb{R}^m$ such that at least two monomials in F are minimal at P. This is also called the *double-min locus* of F.

The Tropical Line

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is the *tropical line* pictured below:

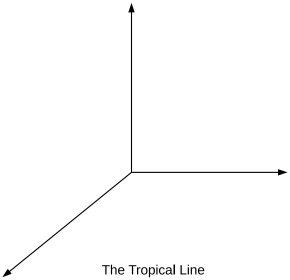

Figure: The tropical line.

Tropicalization and its Discontents

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However, in classical geometry, two distinct lines intersect at a point. As the picture below demonstrates, this isn't always the case in tropical geometry! We're going to want a different definition of a tropical variety.

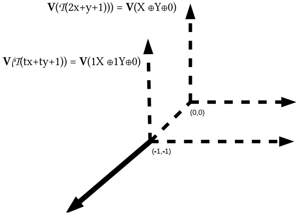

Figure: Two tropical lines intersecting at a ray.

Puiseux Series and the Degree Map

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The field of Puiseux series is the field $K = \mathbb{C}\{\{t\}\}$ of formal power series $a = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $a_1 < a_2 < a_3 < \cdots$ are rational numbers that have a common denominator.

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The only case when this addition relation is not true is when a and b have the same degree, and the coefficients of the leading terms cancel.

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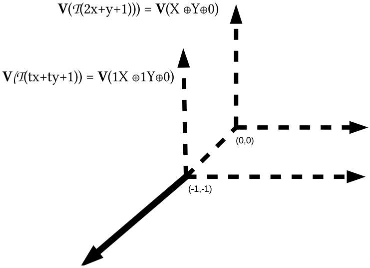
We define a tropical variety in terms of a variety over $\tilde{K}$. Precisely, a *tropical variety* is the image of a variety in $(\tilde{K}^*)^m$ under the degree map

$$(p_1, \ldots, p_m) \in (\tilde{K}^*)^m \mapsto (deg(p_1), deg(p_2), \ldots, deg(p_m)) \in \mathbb{R}^m.$$

Tropical Lines Revisited

The example we saw earlier of two tropical lines intersecting along a ray:



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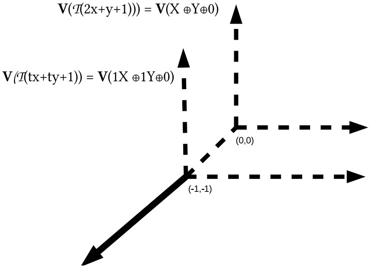

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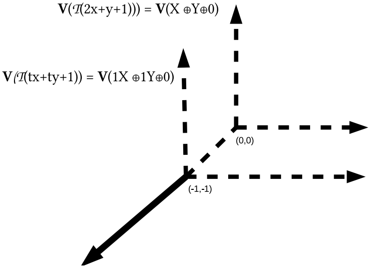

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Given Kapranov's theorem if $I = (f_1, \ldots, f_n)$ then obviously the tropical prevariety determined by the set of tropical polynomials $\{\mathcal{T}(f_1), \ldots, \mathcal{T}(f_n)\}$ contains the tropical variety determined by I:

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If this inequality is an equality, then the set of polynomials $\{f_1, \ldots, f_n\}$ is a *tropical basis* for the ideal they generate.

Outline

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- The rank of a M is the smallest integer r for which M can be written as the sum of r rank one matrices. A matrix has rank one if it is the product of a column vector and a row vector. (The "outer product" of two vectors.)

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What all these terms mean will be explained shortly. For now, we just note that in general

$$\text{tropical rank} \leq \text{Kapranov rank} \leq \text{Barvinok rank}.$$

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The Kapranov Rank

Define a *lift* of an $m \times n$ matrix $A \in \mathbb{R}^{m \times n}$ to be a matrix $\tilde{A} \in \tilde{K}^{m \times n}$ that maps to A under the degree map. The *Kapranov rank* of the matrix A is the minimum rank (defined classically) of any lift of A.

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It's a standard result in algebraic geometry that the $r \times r$ minors of an $m \times n$ matrix of variables are a basis for a prime ideal. The variety corresponding with this prime ideal is called a *determinantal variety*. We can equivalently define the Kapranov rank of a matrix A to be the largest value of r such that A is not in the tropical variety defined by the $r \times r$ minors of an $m \times n$ matrix of variables.

The Tropical Rank

For a square $r \times r$ matrix B we define the tropical determinant to be the obvious analog of its classical counterpart:

$$tropdet(B) := \bigoplus_{\sigma \in S_r} B_{1,\sigma(1)} \odot B_{2,\sigma(2)} \odot \cdots \odot B_{r,\sigma(r)},$$

where the products and sums are tropical, and S_r is the symmetric group on r elements. A square matrix is said to be *tropically singular* if the tropical determinant is realized for more than one permutation.

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the cocircuit matrix of the Fano matroid, has tropical rank three but Kapranov rank four.

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the cocircuit matrix of the Fano matroid, has tropical rank three but Kapranov rank four. In fact, for any cocircuit matrix of a nonrealizable matroid, the tropical rank and Kapranov rank differ. (Note that here we're assuming all our fields have characteristic zero.)

The Big Question

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In the same foundational paper, DSS proved that the $r \times r$ minors *do* form a tropical basis if $r \leq 3$, or if $r = min(m, n)$.

They left open the question of whether there exists a 5×5 matrix with tropical rank three, but Kapranov rank four.

When do the $r \times r$ minors of an $m \times n$ matrix form a tropical basis?

Table: Do the $r \times r$ minors of an $m \times n$ standard matrix form a tropical basis?

$r, min(m,n)$	3	4	5	6	7	8
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4		yes	?	?	no	?
5			yes	?	?	?
6				yes	?	?
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2009 - Chan, Jensen, and Rubei prove the 4×4 minors of a $5 \times n$ matrix form a tropical basis.

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3	yes	yes	yes	yes	yes	yes
4		yes	yes	?	no	?
5			yes	?	?	?
6				yes	?	?
7					yes	?
8						yes

2009 - Chan, Jensen, and Rubei prove the 4×4 minors of a $5 \times n$ matrix form a tropical basis.

When do the $r \times r$ minors of an $m \times n$ matrix form a tropical basis?

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7					yes	?
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2009 - Chan, Jensen, and Rubei prove the 4×4 minors of a $5 \times n$ matrix form a tropical basis. They first ask the more general question about when the $r \times r$ minors of an $m \times n$ matrix form a tropical basis.

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$$\begin{pmatrix} 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 4 & 1 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 1 & 4 & 0 & 0 \end{pmatrix},$$

a 6×6 matrix with tropical rank four but Kapranov rank five.

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2011 - Shitov completes the project. By far the hardest part is proving the 4×4 minors of a $6 \times n$ matrix form a tropical basis.

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We can summarize the answer with the following theorem, known as *Shitov's theorem*.

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Theorem - The $r \times r$ minors of an $m \times n$ matrix form a tropical basis if and only if at least one of the following is true:

1. $r \leq 3$;
2. $r = min(m, n)$;
3. $r = 4$ and $min(m, n) \leq 6$.

Outline

The Symmetric Case

In addition to asking the question for general matrices, in 2009 Chan, Jensen, and Rubei also asked the question of when the $r \times r$ minors of an $n \times n$ *symmetric* matrix form a tropical basis. This is the major question I address in my dissertation.

The Ranks of a Symmetric Tropical Matrix

We begin with the symmetric analogs of the three definitions of rank for tropical matrices we've seen so far.

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- *Symmetric Kapranov* - An $n \times n$ symmetric matrix M has symmetric Kapranov rank r if it is on the tropical variety determined by the $(r+1) \times (r+1)$ minors of a symmetric matrix, but not the $r \times r$ minors.

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- *Symmetric Tropical* - An $n \times n$ symmetric matrix M has symmetric tropical rank r if it is on the tropical prevariety determined by the $(r+1) \times (r+1)$ minors of a symmetric matrix, but not the $r \times r$ minors.

The Symmetric Barvinok Rank

Cartwright and Chan study the symmetric Barvinok rank, along with two additional notions of rank (star tree rank and tree rank) for symmetric matrices, in depth in their paper *Three Notions of Tropical Rank for Symmetric Matrices*.

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The symmetric Barvinok rank of the classical $n \times n$ identity matrix

$$C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is *infinite* for $n \geq 2$.

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is *infinite* for $n \geq 2$. For $n \geq 3$ the matrix C_n has both symmetric tropical and symmetric Kapranov rank three.

Subtleties of Symmetric Kapranov Rank

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For example, The matrix

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is tropically singular, and has a singular lift.

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is tropically singular, and has a singular lift. However, it does not have a *symmetric* singular lift. Try to find one! So, we'd like to say this matrix is nonsingular, and we'll need to modify our definition of singular for symmetric tropical matrices.

An Equivalence Class on S_n

As the example above illustrates, for a square, symmetric matrix it is not enough for two distinct permutations to realize the tropical determinant.

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If $\sigma \in S_n$ has the disjoint cycle decomposition

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$$

then the set of permutations that are *cycle-similar* to σ are the permutations with disjoint cycle decompositions of the form

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and this defines an equivalence class on S_n. If two permutations are not cycle-similar they are *cycle-distinct*.

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and this defines an equivalence class on S_n. If two permutations are not cycle-similar they are *cycle-distinct*.

A symmetric $n \times n$ matrix is symmetrically tropically singular if and only if its tropical determinant is realized by two cycle-distinct permutations.

What's Going On?

The 3×3 symmetric matrix of variables has determinant

$$det \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}$$

$$= x_{1,1}x_{2,2}x_{3,3} + 2x_{1,2}x_{1,3}x_{2,3} - x_{1,1}x_{2,3}^2 - x_{2,2}x_{1,3}^2 - x_{3,3}x_{1,2}^2.$$

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This determinant tropicalizes to the tropical polynomial

$$X_{1,1}X_{2,2}X_{3,3} \oplus X_{1,2}X_{1,3}X_{2,3} \oplus X_{1,1}X_{2,3}^2 \oplus X_{2,2}X_{1,3}^2 \oplus X_{3,3}X_{1,2}^2.$$

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The matrix

$$C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not on the hypersurface the tropical polynomial defines, and so is not singular.

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The matrix

$$C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not on the hypersurface the tropical polynomial defines, and so is not singular. Indeed, its determinant is realized by $(123) = (132)^{-1}$. So, C_3 has tropical rank two, but symmetric tropical rank three.

The Cocircuit Matrix of the Fano Matroid Revisited

A more interesting example of when tropical rank and symmetric tropical rank disagree is the cocircuit matrix of the Fano matroid.

The Cocircuit Matrix of the Fano Matroid Revisited

A more interesting example of when tropical rank and symmetric tropical rank disagree is the cocircuit matrix of the Fano matroid. Indeed, the rows and columns of this matrix can be permuted so as to make it symmetric

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(Thanks Melody Chan for pointing this out to me.)

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(Thanks Melody Chan for pointing this out to me.) However, while this matrix has tropical rank three, its symmetric tropical rank is four! Therefore, it is not an example of a symmetric matrix with symmetric tropical rank three, but greater symmetric Kapranov rank.

When the Minors of a Symmetric Matrix *do* Form A Tropical Basis

So, when *do* the $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis?

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- The case $r = 2$ is trivial.

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- The case $r = n$ is just an application of Kapranov's theorem.
- The case $r = 2$ is trivial.
- The case $r = 3$ is nontrivial, but can be proven by modifying the proof for standard matrices found in Develin, Santos, and Sturmfels.

When the $r \times r$ Minors of a Symmetric Matrix do *not* Form A Tropical Basis

The rows and columns of the 6×6 matrix discovered by Shitov can be permuted so as to make the matrix symmetric:

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 1 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 1 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{pmatrix}.$$

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This matrix has symmetric tropical rank four. As its Kapranov rank is greater than four, a fortiori its symmetric Kapranov rank is greater than four. So, it proves the 5×5 minors of a symmetric 6×6 matrix of variables do *not* form a tropical basis.

The Big One

So, what about the 4×4 minors?

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So, what about the 4×4 minors? If we "pull apart" the cocircuit matrix of the Fano matroid we can construct the following matrix:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

This 13×13 symmetric matrix has symmetric tropical rank three, but greater symmetric Kapranov rank.

When do the $r \times r$ Minors of an $n \times n$ symmetric matrix form a tropical basis?

Table: Do the $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis?

r, n	2	3	4	5	6	7	8	9	10	11	12	13	14
2	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
3		yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
4			yes	?	?	?	?	?	?	?	?	no	no
5				yes	no	no	no	no	no	no	no	no	no
6					yes	no	no	no	no	no	no	no	no
7						yes	no	no	no	no	no	no	no
8							yes	no	no	no	no	no	no
9								yes	no	no	no	no	no
10									yes	no	no	no	no
11										yes	no	no	no
12											yes	no	no
13												yes	no
14													yes

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6					yes	no	no	no	no	no	no	no	no
7						yes	no	no	no	no	no	no	no
8							yes	no	no	no	no	no	no
9								yes	no	no	no	no	no
10									yes	no	no	no	no
11										yes	no	no	no
12											yes	no	no
13												yes	no
14													yes

So, what about $r = 4$ with $5 \leq n \leq 12$?

The 4×4 Minors of a 5×5 Symmetric Matrix of Variables *do* Form a Tropical Basis

As is the case for general matrices, the 4×4 minors of a symmetric 5×5 matrix of variables form a tropical basis.

The 4×4 Minors of a 5×5 Symmetric Matrix of Variables *do* Form a Tropical Basis

As is the case for general matrices, the 4×4 minors of a symmetric 5×5 matrix of variables form a tropical basis. I'll discuss the method used to prove this in my dissertation in the context of the following example:

$$A := \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix},$$

where the blank entries are assumed to be nonnegative, but are otherwise arbitrary. It is easily checked that this matrix has symmetric tropical rank three.

Step One

First, we define the polynomial f_1 to be the determinant

$$f_1 = det \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{1,3} & x_{2,3} & x_{3,3} & x_{3,4} \\ x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} \end{pmatrix},$$

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$$f_1 = det \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{1,3} & x_{2,3} & x_{3,3} & x_{3,4} \\ x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} \end{pmatrix},$$

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and the tropical polynomial F_1 to be its tropicalization, $F_1 = \mathcal{T}(f_1)$. The submatrix A_{55} is on the tropical hypersurface $\mathbf{V}(F_1)$, and therefore, by Kapranov's theorem, there is a symmetric singular lift of A_{55},

$$\tilde{A}_{55} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} \end{pmatrix}.$$

Step Two

Next, we define the polynomial f_2 to be the determinant

$$f_2 = det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & x_{1,5} \\ a_{1,2} & a_{2,2} & a_{2,3} & x_{2,5} \\ a_{1,3} & a_{2,3} & a_{3,3} & x_{3,5} \\ x_{1,5} & x_{2,5} & x_{3,5} & x_{5,5} \end{pmatrix},$$

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and the tropical polynomial F_2 to be its tropicalization, $F_2 = \mathcal{T}(f_2)$. The point $(A_{1,5}, A_{2,5}, A_{3,5}, A_{5,5})$ is on the tropical hypersurface $\mathbf{V}(F_2)$ and so, again by Kapranov's theorem, it lifts to a point $(a_{1,5}, a_{2,5}, a_{3,5}, a_{5,5})$ on f_2.

Step Three

Finally, we define the (linear) polynomial f_3 to be the determinant

$$f_3 = det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} \\ a_{1,5} & a_{2,5} & a_{3,5} & x_{4,5} \end{pmatrix},$$

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Some Preliminary Definitions

For a tropical polynomial F the monomials contained by F that are minimal at a point P are called the *minimizing monomials* of F at P.

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$$X_\sigma = X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)}.$$

Suppose $A \in \mathbb{R}^{n \times n}$ is a symmetric $n \times n$ matrix, and X is a symmetric $n \times n$ matrix of variables. For any submatrix of A there is a corresponding submatrix of X, and the determinant of this submatrix of X is a polynomial. The submatrix of A determines a set of minimizing monomials in this determinant.

The Method of Joints

With these definitions in hand we're ready to formally define joints. Suppose A is a symmetric matrix, and there are distinct indices i and j (assume without loss of generality $i < j$) such that:

- The principal submatrix A_{ii} is symmetrically tropically singular, and there are distinct minimizing monomials $X_{\sigma_1}, X_{\sigma_2}$, such that the variables in X_{σ_1} involving the index j are not the same as the variables in X_{σ_2} involving the index j.

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- The same is true with i and j reversed.
- The submatrix A_{ji} is symmetrically tropically singular, and there are two minimizing monomials X_{τ_1}, X_{τ_2} such that X_{τ_1} contains the variable $X_{i,j}$, while X_{τ_2} does not.

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The indices i and j are *joints* of the matrix A. If the submatrix A_{ii} satisfies the first condition above, we say it *satisfies the joint requirement* for joints i and j. Similarly for the submatrix A_{jj}.

If a 5×5 symmetric matrix has joints, then it has symmetric Kapranov rank at most three.

Symmetric Tropical Rank Three

Theorem - The 4×4 minors of a symmetric 5×5 matrix form at tropical basis.

Symmetric Tropical Rank Three

Theorem - The 4×4 minors of a symmetric 5×5 matrix form at tropical basis.

This theorem is proved by demonstrating that, with one exception, every 5×5 symmetric matrix with symmetric tropical rank three has joints. The exception is dealt with separately, and proven to also have a symmetric rank three lift.

When do the $r \times r$ Minors of an $n \times n$ symmetric matrix form a tropical basis?

Table: Do the $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis?

r, n	2	3	4	5	6	7	8	9	10	11	12	13	14
2	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
3		yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
4			yes	?	?	?	?	?	?	?	?	no	no
5				yes	no	no	no	no	no	no	no	no	no
6					yes	no	no	no	no	no	no	no	no
7						yes	no	no	no	no	no	no	no
8							yes	no	no	no	no	no	no
9								yes	no	no	no	no	no
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6					yes	no	no	no	no	no	no	no	no
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I suspect that each of the question marks is, in fact, a yes.

Outline

Large Prevarieties

When two tropical lines intersect along a ray they don't just fail to be a tropical variety, they fail big!

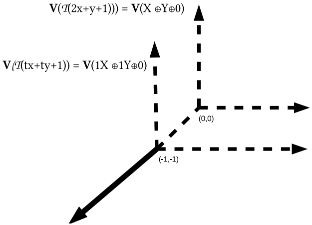

Figure: Two tropical lines intersecting at a ray.

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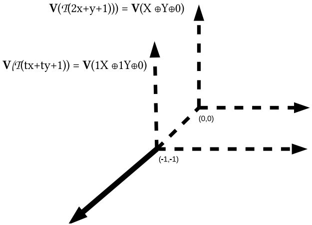

Figure: Two tropical lines intersecting at a ray.

That is to say, the tropical prevariety determined by the lines above does not just contain the tropical variety. The tropical prevariety is in fact of greater dimension than the tropical variety.

Tropical Bases are *Not* Determined by Dimension

It is not the case, in general, that if a basis fails to be a tropical basis then its corresponding tropical prevariety has greater dimension than its corresponding tropical variety.

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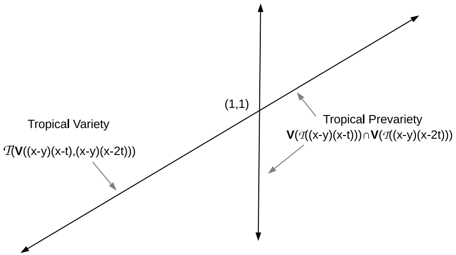

Figure: A connected example of a basis that is not a tropical basis, but in which both the tropical variety and tropical prevariety have the same dimension.

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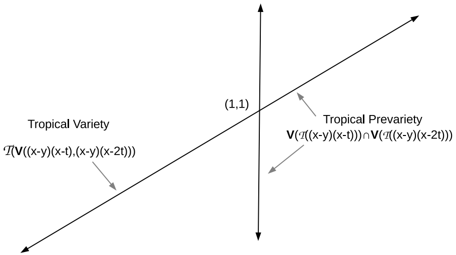

Figure: A connected example of a basis that is not a tropical basis, but in which both the tropical variety and tropical prevariety have the same dimension.

(Thanks Brian Osserman for showing me this example.)

Determinantal Prevarieties Fail Big

In my dissertation I prove that, for determinantal ideals, it is the case that if the minors fail to be a tropical basis they fail big.

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Theorem - The $r \times r$ minors of an $m \times n$ matrix do not form a tropical basis if and only if the dimension of the tropical prevariety determined by the minors is greater than the dimension of the tropical variety determined by the minors.

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Theorem - The $r \times r$ minors of an $m \times n$ matrix do not form a tropical basis if and only if the dimension of the tropical prevariety determined by the minors is greater than the dimension of the tropical variety determined by the minors.

Note that I think this result was already known to the mathematical community, but I'm unaware of a proof outside of my dissertation.

Symmetric Determinantal Prevarieties Usually Fail Big

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Theorem - If $r > 4$ the $r \times r$ minors of a symmetric $n \times n$ matrix do not form a tropical basis if and only if the dimension of the tropical prevariety determined by the minors is greater than the dimension of the tropical variety determined by the minors.

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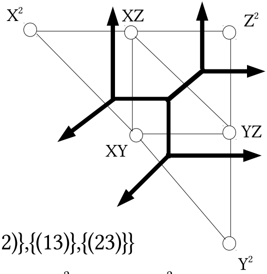
Theorem - If $r > 4$ the $r \times r$ minors of a symmetric $n \times n$ matrix do not form a tropical basis if and only if the dimension of the tropical prevariety determined by the minors is greater than the dimension of the tropical variety determined by the minors.

For $r = 4$ the question remains unanswered. I suspect for $r = 4$ and $n = 13$ the dimension of the prevariety and variety are the same.

The Seed Blooms

Finally, we return to the observation made at the beginning of this talk. Namely, that the conic pictured below is nonsingular.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



detprof(A) = {{(123)},{(12)},{(13)},{(23)}}

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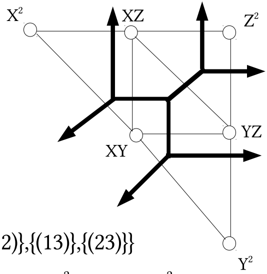

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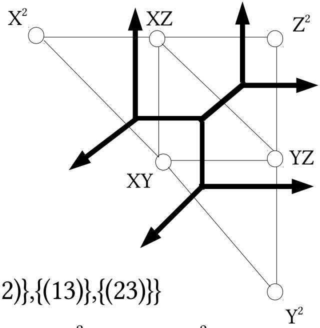
But now there is no mystery. The symmetric tropical matrix corresponding with this conic is tropically singular, but it is not symmetrically tropically singular, which is the important criterion.

Tropical Quadrics and Their Dual Complexes

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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It is worth noting that, for tropical conics, the dual complex of the conic is completely determined by the cycle-similar permutation classes that realize the determinant of the matrix, and the classes that realize the determinant of the principal submatrices.

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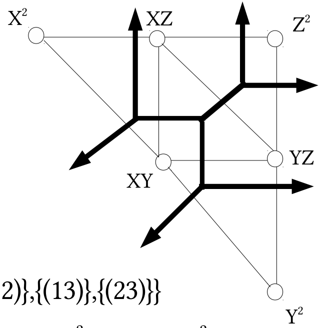

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It is worth noting that, for tropical conics, the dual complex of the conic is completely determined by the cycle-similar permutation classes that realize the determinant of the matrix, and the classes that realize the determinant of the principal submatrices. I believe this generalizes to all tropical quadrics.

Thank You!

"City," he cried, and his voice rolled over the metropolis like thunder, "I am going to tropicalize you." - Salman Rushdie, **The Satanic Verses**