

TROPICAL QUADRICS

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In this quick talk I'll introduce the basic ideas behind tropical geometry and its particular application to the study of tropical quadrics. I'll then discuss some of the combinatorial aspects of tropical quadrics, and some questions that can be asked about them.

TROPICAL ALGEBRA, TROPICAL GEOMETRY, AND PUISEUX SERIES

Tropical algebra begins with the tropical min-plus¹ semiring, which is a semiring defined over the set of real numbers with the operations

$$a \oplus b = \min(a, b),$$

$$a \otimes b = a + b.$$

So, for example, $5 \oplus 3 = 3$, while $5 \otimes 3 = 8$. This is a semiring,² and not a ring, because addition is not invertible. If we're given $a \oplus 3 = 3$, we don't know a exactly. All we know is $a \geq 3$.

Tropical geometry is an attempt to do algebraic geometry in the tropical semiring. We define a tropical hypersurface as the “double-min locus” of a polynomial in the tropical semiring. That is to say, the set of values for which at least two monomials are simultaneously minimized. So, for example, the tropical hypersurface associated with the tropical linear polynomial

$$X \oplus Y \oplus 0$$

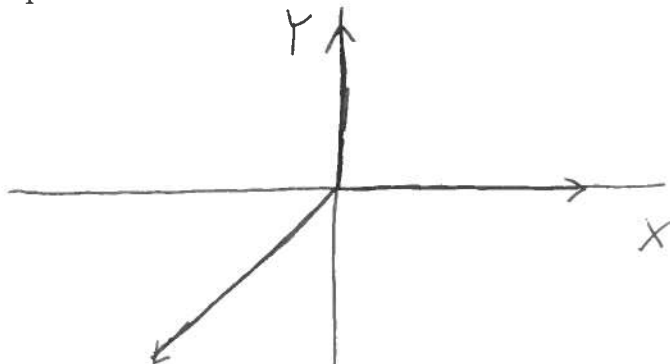
is the set of values:

$$\begin{cases} X = Y & X, Y \leq 0 \\ X = 0 & Y \geq 0 \\ Y = 0 & X \geq 0 \end{cases}$$

¹You can also use a max-plus semiring, and all the results are, mutatis mutandis, the same. Some authors use min, some use max, and there's a bit of a VHS vs. Betamax war going on in the published papers right now. As of this writing, a clear winner has yet to emerge.

²Semifield, actually.

This “tropical line” looks like this



Now, why does this double-min locus definition make sense? Well, one way to look at it is to first start with standard algebraic geometry over the field of *Puiseux series*. The field of Puiseux series, K , is the field of all formal series

$$c_0t^{a_0} + c_1t^{a_1} + \dots$$

where t is a variable, the c_i terms are from an algebraically closed field k (usually taken to be \mathbb{C}), $c_0 \neq 0$, and the sequence a_0, a_1, \dots is an increasing sequence of rational numbers, where eventually the denominators stabilize. Puiseux proved (thus the name) that if k has characteristic zero then the field of Puiseux series is algebraically closed.

We can define a valuation on K as

$$\text{val}(c_0t^{a_0} + c_1t^{a_1} + \dots) = a_0.$$

Let's look at the valuation of the line

$$X + Y = 1$$

defined over the field of Puiseux series. Well, for the above equality to be true, we must have either that $\text{val}(X) = \text{val}(Y) \leq 0$, or $0 = \text{val}(X) \leq \text{val}(Y)$, or $0 = \text{val}(Y) \leq \text{val}(X)$. In other words, the image of this line under the valuation map will be exactly the tropical line we examined above! This idea generalizes, and this connection is fundamental to tropical geometry. Tropical geometry is not only interesting by itself, but it can also tell us things about regular algebraic geometry, and vice-versa, through the connection just outlined.

TROPICAL QUADRICS

In regular algebraic geometry a quadric in \mathbb{P}^n is the variety defined by a degree 2 polynomial in $n+1$ variables. If we assume the characteristic of our field is not 2 then any quadric:

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + \cdots + 2a_{n(n+1)}x_nx_{n+1} + a_{(n+1)(n+1)}x_{n+1}^2,$$

can be written as

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ a_{12} & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

This gives us a bijective correspondence between quadrics in \mathbb{P}^n and symmetric $(n+1) \times (n+1)$ matrices. A quadric is singular if its corresponding symmetric matrix is singular, and the rank of a quadric is the rank of its corresponding symmetric matrix.

For tropical quadrics in \mathbb{TP}^n the same bijection exists, except we don't have to deal with those annoying 2s in the coefficients of the degree 2 polynomial defining the quadric.

Before we get any farther we need to address one situation. Suppose we have a tropical quadric in \mathbb{TP}^1 . This will correspond to a polynomial of the form:

$$aX^2 \oplus bXY \oplus cY^2.$$

We note that if $ac < b^2$,³ then the monomial XY will never be minimized. We want to deal with tropical quadrics where “every monomial has its day”, and so we'll always require that $a_{ij}^2 < a_{ii}a_{jj}$.

Now that's out of the way, a question that naturally comes up here is, what does it mean for an $n \times n$ matrix $A = (a_{ij})$ to be singular in tropical geometry. Well, if we tropicalize the determinant we get:

$$\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)},$$

³This is in the tropical semiring. So, $ac < b^2$ translates into $a + c < 2b$ in standard arithmetic.

where the sums and products are tropical, and we call a matrix singular if the above sum is minimized for two distinct permutations.⁴

Now, our first guess as to what it means for a tropical quadric to be singular would probably be that its corresponding symmetric matrix is singular in the above sense. However, this doesn't quite work. Let's take a look at a few examples so we can get an intuitive idea why.

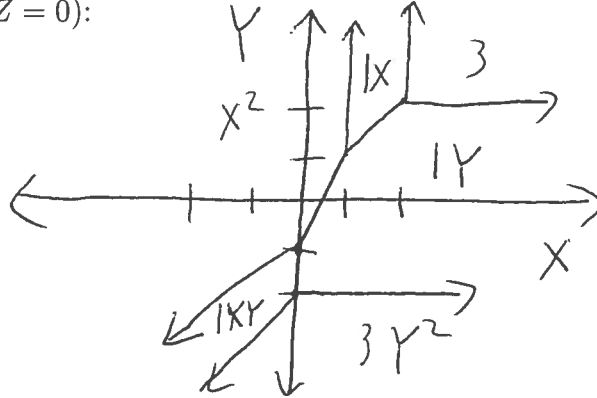
First, let's look at the quadric defined by the tropical polynomial:

$$X^2 \oplus 1XY \oplus 1XZ \oplus 3Y^2 \oplus 1YZ \oplus 3Z^2.$$

Its corresponding symmetric matrix is:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

This matrix is uniquely minimized by the permutation (1)(23), so the matrix is nonsingular. The corresponding tropical curve looks like (setting $Z = 0$):



On the other hand, the quadric defined by the tropical polynomial:

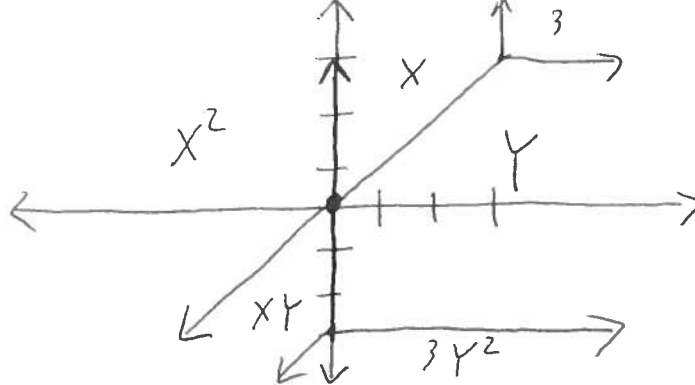
$$X^2 \oplus XY \oplus XZ \oplus 3Y^2 \oplus YZ \oplus 3Z^2$$

has the corresponding symmetric matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

⁴Note the standard $(-1)^{\text{sgn}(\sigma)}$ in the determinant goes away when we tropicalize. So, the tropical determinant and the tropical permanent are the same thing.

This matrix has minimizing permutations (1)(23), (123), and (132), so it's singular. The corresponding tropical curve looks like:



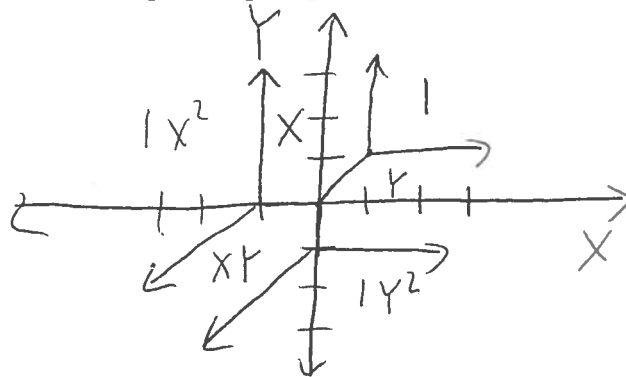
However, things get a little tricky if we look at the tropical quadric defined by the tropical polynomial:

$$1X^2 \oplus XY \oplus XZ \oplus 1Y^2 \oplus YZ \oplus 1Z^2.$$

The corresponding symmetric matrix is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix has minimizing permutations (123) and (132), and so is singular. The corresponding tropical curve looks like:



That curve... doesn't look singular. And, in fact, under any reasonable definition of what singular means for tropical curves, it's not. So, what do we do? Well, we need to modify, for our purposes, the definition of what it means for a symmetric matrix to be singular.

First, we note $(123) = (132)^{-1}$. This isn't a coincidence. What we want to do is define an equivalence class on permutations. If a permutation σ has cycle decomposition

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_m,$$

where the σ_i are cycles, then the *symmetric permutation class* of σ is the set of all permutations of the form:

$$\sigma_1^{\pm 1} \sigma_2^{\pm 1} \cdots \sigma_m^{\pm 1}.$$

So, for example, (123) and (132) are in the same symmetric permutation class. So are (123)(456), (132)(456), (123)(465), and (132)(465). We say two permutations are *cycle-similar* if they're in the same symmetric permutation class, and they're *cycle-distinct* otherwise.

For symmetric matrices in our context, we define a symmetric matrix as being *symmetrically singular* if its tropical determinant is realized by two cycle-distinct permutations.

Now, why does this definition make sense? Well, it turns out that while there is a singular 3×3 matrix over the field of Puiseux series that tropicalizes to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

there is no *symmetric* singular 3×3 matrix over the field of Puiseux series that tropicalizes to it. And, in general, it will be the case that a singular symmetric matrix over the Puiseux series tropicalizes to a given symmetric tropical matrix if and only if that tropical matrix is symmetrically singular.

Already there are some interesting and unexpected results here. For example, it's pretty easy to prove that if the determinant of a symmetric matrix is realized by the permutation

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_m,$$

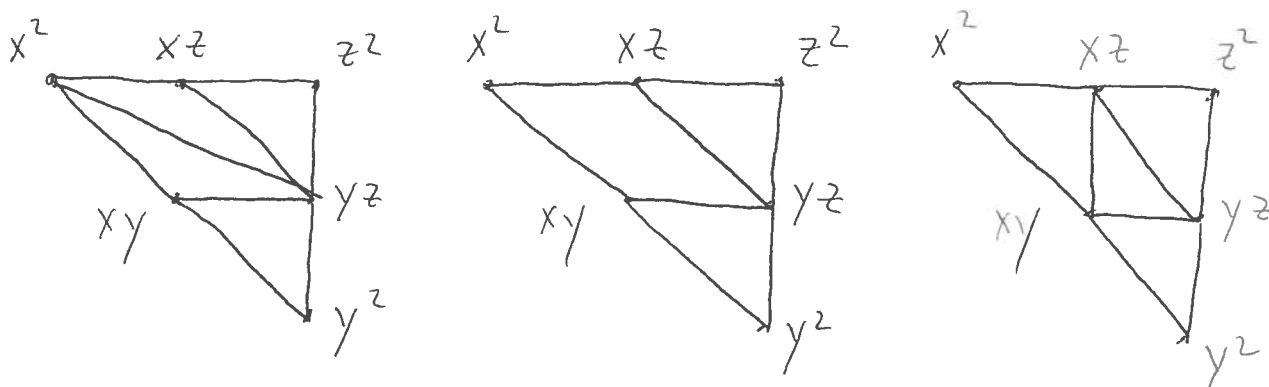
where the σ_i s are cycles, then if any σ_j is an odd-cycle that is larger than a transposition the matrix must be symmetrically singular.

DUAL COMPLEXES AND THEIR COMBINATORICS

A tropical hypersurface is a polyhedral complex, and for this polyhedral complex we can define its dual complex. Abstractly, the dual complex is a polyhedral complex with vertices corresponding to the

monomials appearing in the polynomial, and polyhedrons corresponding to the convex hulls of vertices that can be simultaneously minimized.

So, the dual complexes for the three example quadrics we've studied so far are:

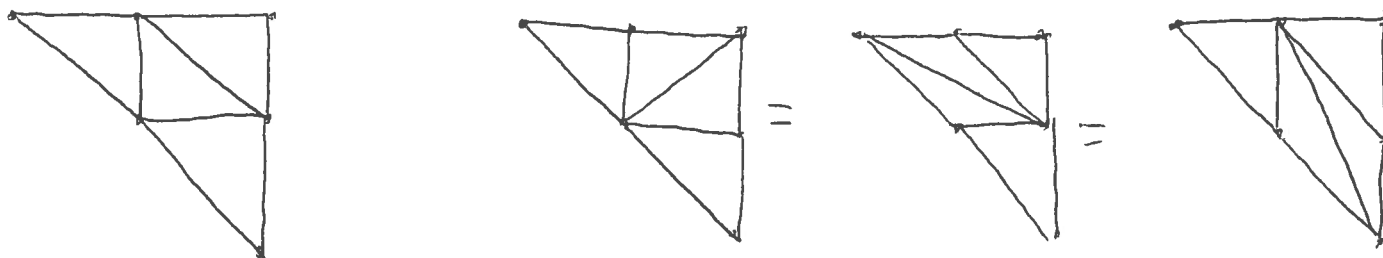


Note that, in all three cases, the dual complex is completely determined by the symmetric permutation classes of the permutations that minimize the determinant of the symmetric matrix determined by the quadric.

We believe this idea generalizes. More precisely, we believe the dual complex of a tropical quadric is completely determined by the symmetric permutation classes of the permutations that realize the determinant of the symmetric matrix corresponding to the tropical quadric, and by the same data for all the principle submatrices of this matrix.

Another natural question we can ask is how many combinatorial types of tropical quadrics are there? In other words, how many dual complexes are there? Perhaps more interesting, we can say two dual complexes are the same if one can be obtained from the other just by a relabeling of the variables. If we restrict ourselves to just studying quadrics that are not only nonsingular, but for which every principle submatrix of the corresponding symmetric matrix is symmetrically nonsingular, we can ask the same question.

In the latter case for quadrics in \mathbb{TP}^2 the answer is 2. These guys:



We've determined computationally that for quadrics in \mathbb{TP}^3 the answer is 15. It's an open and interesting question what the number is for quadrics in \mathbb{TP}^n for $n > 3$.