

Math 2280 - Lecture 7: Population Models

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Today we're going to explore one of the major applications of differential equations - population models. We'll also explore these models tomorrow in the context of autonomous differential equations.

The exercises for section 2.1 are:

Section 2.1 - 1, 8, 11, 16, 29

Population Models

Population growth models depends upon two parameters - birth and death. We call β the "birth rate" of the population, and δ the "death rate" of the population. The population model is

$$\frac{dP}{dt} = (\beta - \delta)P.$$

The terms β and δ are not necessarily constants, and could themselves be functions of time or the population size.

The simplest population model is one in which β and δ are constant. In this scenario the rate of growth of the population is directly proportional to the population. A population with this characteristic is modeled by the differential equation

$$\frac{dP}{dt} = kP.$$

There are many things in life that actually do fit this model¹, and it can be a reasonable short-term model for population growth. However, the thing about exponentials is they grow really, really fast, and no real population can keep growing exponentially forever. Something, usually the exhaustion of resources, enters the picture to correct things.

So, let's look at a slightly more sophisticated population model. Suppose the birth rate of our population is a *linear* decreasing function of the population size. So,

$$\beta = \beta_0 - \beta_1 P,$$

while the death rate is constant,

$$\delta = \delta_0.$$

In this case the population model will be

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P,$$

which we can rewrite as

$$\frac{dP}{dt} = aP - bP^2.$$

If the parameters a and b are both positive then the above equation is called the *logistic equation*. It's actually even more convenient to rewrite it as

$$\frac{dP}{dt} = kP(M - P)$$

where $k = b$ and $M = a/b$.

¹Compound interest, for example.

Example - Solve the logistic growth equation with initial condition $P(0) = P_0 > 0$.

Solution - The differential equation is:

$$\frac{dP}{dt} = kP(M - P).$$

This is a separable differential equation, and we can rewrite it as:

$$\frac{dP}{P(M - P)} = kdt.$$

In order to integrate the left side we want to do a partial fraction decomposition. Doing this gives us:

$$\frac{\frac{1}{M}dP}{P} + \frac{-\frac{1}{M}dP}{M - P} = kdt.$$

If we multiply both sides by M we get:

$$\frac{dP}{P} - \frac{dP}{M - P} = Mkdt.$$

Integrating both sides gives us:

$$\ln P - \ln(M - P) = Mkt + C.$$

We note $\ln P - \ln(M - P) = \ln\left(\frac{P}{M - P}\right)$. Using this and solving for P gives us (after some algebra):

$$P(t) = \frac{MC}{1 + Ce^{-Mkt}}.$$

If we plug in $P(0) = P_0$ and solve for C we get, again after a little algebra:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}.$$

This is the solution to the logistic growth equation.

Consider a population $P(t)$ of unsophisticated animals in which females rely solely on chance encounters to meet males for reproductive purposes.² Here the rate of growth will be proportional to the product of the number of males, $P/2$, and the number of females, $P/2$. So, births will occur at a rate kP^2 , and the “birth rate” will be kP . If the death rate, δ , is constant then the differential equation modeling our population will be

$$\frac{dP}{dt} = kP^2 - \delta P = kP(P - M).$$

This type of differential equation exhibits very interesting behavior.

Example - Solve the differential equation above, and explain why it’s sometimes called the “Doomsday-Extinction” model.

Solution - The differential equation is:

$$\frac{dP}{dt} = kP(P - M).$$

This is a separable differential equation, and we can rewrite it as:

$$\frac{dP}{P(P - M)} = kdt.$$

As in the previous example, we do a partial fraction decomposition and integrate both sides to get:

$$\int \left(\frac{1}{P - M} - \frac{1}{P} \right) dp = \int Mkdt.$$

Taking these integrals and using some basic properties of logarithms we get:

²For example, bars on weekends...

$$\ln\left(\frac{P-M}{P}\right) = Mkt + C.$$

Doing some algebra we can solve this for P to get:

$$P(t) = \frac{M}{1 - Ce^{Mkt}}.$$

Plugging in $P(0) = P_0$ and solving for C we get, again after some algebra:

$$P(t) = \frac{MP_0}{P_0 - (P_0 - M)e^{Mkt}}.$$

If $P_0 < M$ then $\lim_{t \rightarrow \infty} P(t) = 0$. So, extinction.

If $P_0 > M$ then $\lim_{t \rightarrow t_d^-} P(t) = \infty$, where $t_d = \frac{\ln\left(\frac{P_0}{P_0 - M}\right)}{Mk}$. This number t_d is “Doomsday”, the finite amount of time before we have a vertical asymptote.

So, the model is called “Doomsday-extinction” because, unless $P_0 = M$ we get either extinction or “Doomsday”.