Math 2280 - Lecture 7: Population Models

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Today we're going to explore one of the major applications of differential equations - population models. We'll also explore these models tomorrow in the context of autonomous differential equations.

The exercises for section 2.1 are:

$$\text{Section 2.1 - 1, 8, 11, 16, 29}$$

Population Models

Population growth models depends upon two parameters - birth and death. We call β the "birth rate" of the population, and δ the "death rate" of the population. The population model is

$$\frac{dP}{dt} = (\beta - \delta)P.$$

The terms β and δ are not necessarily constants, and could themselves be functions of time or the population size.

The simplest population model is one in which β and δ are constant. In this scenario the rate of growth of the population is directly proportional to the population. A population with this characteristic is modeled by the differential equation

$$\frac{dP}{dt} = kP.$$

There are many things in life that actually do fit this model[1], and it can be a reasonable short-term model for population growth. However, the thing about exponentials is they grow really, really fast, and no real population can keep growing exponentially forever. Something, usually the exhaustion of resources, enters the picture to correct things.

So, let's look at a slightly more sophisticated population model. Suppose the birth rate of our population is a *linear* decreasing function of the population size. So,

$$\beta = \beta_0 - \beta_1 P,$$

while the death rate is constant,

$$\delta = \delta_0.$$

In this case the popluation model will be

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P,$$

which we can rewrite as

$$\frac{dP}{dt} = aP - bP^2.$$

If the parameters a and b are both positive then the above equation is called the *logistic equation*. It's actually even more convenient to rewrite it as

$$\frac{dP}{dt} = kP(M - P)$$

where $k = b$ and $M = a/b$.

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[1]Compound interest, for example.

Example - Solve the logistic growth equation with initial condition $P(0) = P_0 > 0$.

Consider a population $P(t)$ of unsophisticated animals in which females rely solely on chance encounters to meet males for reproductive purposes.[2] Here the rate of growth will be proportional to the product of the number of males, $P/2$, and the number of females, $P/2$. So, births will occur at a rate kP^2, and the "birth rate" will be kP. If the death rate, δ, is constant then the differential equation modeling our population will be

$$\frac{dP}{dt} = kP^2 - \delta P = kP(P - M).$$

This type of differential equation exhibits very interesting behavior.

Example - Solve the differential equation above, and explain why it's sometimes called the "Doomsday-Extinction" model.

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[2]For example, bars on weekends...

More space for the example problem.