Math 2280 - Lecture 6: Substitution Methods for First-Order ODEs and Exact Equations

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In today's lecture we're going to examine another technique that can be useful for solving first-order ODEs. Namely, substitutuion. Now, as with u-substitutuion from calculus, figuring out the right substitution to make, assuming there even is a right one, is not always obvious. It's part science, and part art. However, there are a few general forms that we can recognize, and for which there are straightforward methods. We'll discuss some of these today.

We will also learn about another special type of differential equation, an exact equation, and how these can be solved.

The exercises for this section are:

Section 1.6 - 1, 3, 13, 16, 22, 26, 31, 36, 56

The Idea of Substitution

Suppose we're given the first-order differential equation in standard form that we've by now all learned to know and love:

$$\frac{dy}{dx} = f(x, y).$$

The function $f(x, y)$ may contain a combination of the variables x and y:

$$v = \alpha(x, y)$$

that suggests itself as a new independent variable v.

If we can solve $v = \alpha(x, y)$ for y in terms of x and v:

$$y = \beta(x, v),$$

then the chain rule gives us

$$\frac{dy}{dx} = \beta_x + \beta_v \frac{dv}{dx}.$$

Using this, and the relation

$$\frac{dy}{dx} = f(x, y)$$

we can get a *new* differential equation,

$$\frac{dv}{dx} = g(x, v),$$

which we might be able to solve for v, and from there we can solve for y. No problem, right?[1]

[1] Yeah, it probably looks a little scary now. It will (hopefully) become more clear and less frightening once you've worked through some examples.

Example - Find a general solution to the differential equation

$$yy' + x = \sqrt{x^2 + y^2}.$$

Solution - If we make the substitution

$$v = x^2 + y^2$$

then its derivative is

$$\frac{dv}{dx} = 2x + 2y\frac{dy}{dx} = 2x + 2yy'.$$

We can use the starting differential equation to derive the substitution

$$y' = \frac{\sqrt{v}}{y} - \frac{x}{y}$$

and using this substitutuion to solve for $\frac{dv}{dx} = v'$ we get:

$$v' = 2\sqrt{v}.$$

This is a separable differential equation, and we can rewrite it as:

$$\frac{dv}{\sqrt{v}} = 2dx.$$

Integrating both sides gives us:

$$2\sqrt{v} = 2x + C.$$

Solving this for v, and playing a little fast and loose with the unknown constant C, we get:

$$v = (x + C)^2.$$

Noting $v = x^2 + y^2$ we get the curve:

$$x^2 + y^2 = (x + C)^2.$$

Homogeneous Equations

A *homogeneous* first-order differential equation is one that can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

If we make the substitutuion $v = y/x$ then we can transform our equation into a separable equation

$$x\frac{dv}{dx} = F(v) - v.$$

Example - Find the general solution to the differential equation

$$(x + y)y' = x - y.$$

Solution - We can rewrite this differential equation as:

$$y' = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}.$$

This is a homogeneous equation, and so making the substitution $v = \frac{y}{x}$ we get:

$$x\frac{dv}{dx} = \frac{1 - v}{1 + v} - v.$$

We can rewrite this equation as:

$$\frac{1 + v}{1 - 2v - v^2}dv = \frac{1}{x}dx.$$

In order to take the integral on the left side,

$$\int \frac{1+v}{1-2v-v^2}dv,$$

we make the substitution $u = 1 - 2v - v^2$, and $du = (-2-2v)dv$. Upon this substitution our integral becomes

$$-\frac{1}{2}\int \frac{du}{u} = \ln\left(\frac{1}{\sqrt{u}}\right).$$

The integral on the right side above is $\int \frac{1}{x}dx = \ln x + C$. So, we get the equality:

$$\ln\left(\frac{1}{\sqrt{u}}\right) = \ln x + C,$$

which we can rewrite as:

$$\frac{1}{\sqrt{u}} = Cx.$$

We can rewrite this as:

$$1 = Cx^2u.$$

Substituting $u = 1 - 2v - v^2$ and $v = \frac{y}{x}$ we get:

$$1 = Cx^2\left(1 - 2\frac{y}{x} - \frac{y^2}{x^2}\right) = C(x^2 - 2xy - y^2).$$

We can rewrite this equality, playing fast and loose with unknown constants as always, as:

$$C = x^2 - 2xy - y^2.$$

Bernoulli Equations

A Bernoulli equation[2] is a first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

If $n = 0$ or $n = 1$ then it's just a linear differential equation. Otherwise, if we make the substitution

$$v = y^{1-n}$$

the differential equation above transforms into the linear equation

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which we can then solve.

[2]Named after one member of that famous 17th-century mathematical family. I'm not sure which.

Example - Find the general solution to the differential equation

$$xy' + 6y = 3xy^{4/3}.$$

Solution - If we divide the above equation by x we get:

$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{\frac{4}{3}}.$$

This is a Bernoulli equation with $n = \frac{4}{3}$. So, if we make the substitution $v = y^{-\frac{1}{3}}$ the equation transforms into:

$$\frac{dv}{dx} - \left(\frac{1}{3}\right)\left(\frac{6}{x}\right)v = \left(-\frac{1}{3}\right)3.$$

This simplifies to:

$$\frac{dv}{dx} - \frac{2}{x}v = -1.$$

This is a first-order linear differential equation. The integrating factor will be:

$$\rho = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2},$$

and using this integrating factor we get the equality:

$$\frac{d}{dx}\left(\frac{v}{x^2}\right) = -\frac{1}{x^2}.$$

Integrating both sides we get:

$$\frac{v}{x^2} = \frac{1}{x} + C.$$

Solving for v we get:

$$v = Cx^2 + x.$$

Plugging back in $v = y^{-\frac{1}{3}}$ and solving for y gives us:

$$y(x) = \frac{1}{(Cx^2 + x)^3}.$$

Exact Differential Equations

We've seen in our solutions to differential equations that sometimes, frequently even, the solution is not an explicit equation describing y as a function of x, but is instead an implicit function of the form

$$F(x, y) = C,$$

where the dependence of y on x is implicit. We can recover our initial differential equation by differentiating both sides with respect to x, and then solving for dy/dx:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0;$$

solving for dy/dx:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

We can write the first equation above a bit more symmetrically as

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0.$$

Turning this around, suppose we're given a differential equation in the form

$$M(x, y)dx + N(x, y)dy = 0.$$

If the functions $M(x, y)$ and $N(x, y)$ are such that there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y),$$

and

$$\frac{\partial F}{\partial y} = N(x, y),$$

then the (implicit) solution to the differential equation will be

$$F(x, y) = C.$$

Recall from multivariable calculus that if the mixed partial derivatives F_{xy} and F_{yx} are continuous in a open subset of the xy-plane, then they're equal on that subset. In practice F_{xy} and F_{yx} are usually continuous, and so if $F_x = M$ and $F_y = N$ then we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This is a *necessary* condition. Turns out it's also sufficient. That is to say if we have the equality above, and both partial derivatives are continuous, then our desired function $F(x, y)$ exists.

Example - Solve the differential equation

$$(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0.$$

Solution - First we check that it's exact:

$$\frac{\partial}{\partial y}(2xy^2 + 3x^2) = 4xy,$$
$$\frac{\partial}{\partial x}(2x^2y + 4y^3) = 4xy.$$

So, it's exact. That checks out. Now, to find $F(x, y)$ we take the integral of our function M with respect to x:

$$\int (2xy^2 + 3x^2)dx = x^2y^2 + x^3 + g(y).$$

Here $g(y)$ is an unknown function of y. To solve for $g(y)$ we take the derivative of the function with respect to y and set it equal to the function N to get:

$$\frac{\partial}{\partial y}(x^2y^2 + x^3 + g(y)) = 2x^2y + g'(y) = 2x^2y + 4y^3.$$

So, $g'(y) = 4y^3$, which means $g(y) = y^4$, and our function $F(x, y)$ is:

$$F(x, y) = x^2y^2 + x^3 + y^4.$$

So, the solution to the ODE is:

$$x^2y^2 + x^3 + y^4 = C.$$