Math 2280 - Lecture 5: Linear First-Order Equations

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Today we're going to examine the first-order version of a type of differential equation that we're going to see quite a bit more of this summer. So, get comfortable with them.

This type of differential equation is a *linear* differential equation. A first-order linear differential equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Today, we're going to learn how to solve differential equations of this form.

The exercises for this section are:

Section 1.5 - 1, 15, 21, 29, 38, 42

First-Order Linear Differential Equations

When we say a differential equation is *linear*, we mean it's linear in the dependent variable y and its derivatives. So, the equation

$$y' + (e^x \sin x^2)y = x^3 + 2x^2 - 5x + 2$$

is linear, while the differential equation

$$(y')^2 = x$$

is not.

If we have a first-order linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we can multiply both sides by an *integrating factor*. An integrating factor is a function $\rho(x, y)$ such that, if we multiply both sides by that function, we can recognize both sides of the equation as a derivative. In this case the integrating factor is

$$\rho(x) = e^{\int P(x)dx}.$$

The derivative of ρ is[1]

$$\frac{d\rho}{dx} = P(x)e^{\int P(X)dx}.$$

Using this, we see that the derivative of $ye^{\int P(x)dx}$ is

$$\frac{d}{dx}(ye^{\int P(x)dx}) = e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y.$$

[1] That's the sound of the men working on the chain... rule.

Using this, we see that if we have the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we can multiply both sides by the integrating factor $\rho(x) = e^{\int P(x)dx}$ to get

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x).$$

If we then integrate both sides with respect to x we get

$$e^{\int P(x)dx}y = \int (e^{\int P(x)dx}Q(x))dx + C,$$

which we can then solve for y to get:

$$y(x) = e^{-\int P(x)dx}\left(\int (e^{\int P(x)dx}Q(x))dx + C\right).$$[2]

Daaaaang! Let's do an example.

[2]The book warns you to *not* memorize this equation. So, whatever you do, don't go memorizing this equation. You should just memorize the method by which we derived the equation. Or, I suppose, in a pinch you could also memorize the equation. But, in practice (at least in this class), things usually aren't as scary as this general solution might make them look.

Example - Solve the initial value problem

$$y' - 2xy = e^{x^2} \qquad y(0) = 0.$$

Solution - The integrating factor will be:

$$\rho(x) = e^{-\int 2x dx} = e^{-x^2}.$$

Multiplying both sides of the differential equation by ρ we get:

$$e^{-x^2}y' - 2xe^{-x^2}y = 1.$$

Integrating both sides we get:

$$\int \frac{d}{dx}(e^{-x^2}y)dx = \int dx,$$
$$\Rightarrow e^{-x^2}y = x + C.$$

So, we get:

$$y(x) = Ce^{x^2} + xe^{x^2}.$$

Plugging in the initial condition $y(0) = 0$ we get:

$$y(0) = Ce^{0^2} + 0e^{0^2} = C = 0.$$

So, the solution to the initial value problem is:

$$y(x) = xe^{x^2}.$$

Existence, Uniqueness, and Examples

Now, again, before we spend too long trying to solve a differential equation, we'd like to know whether or not a solution even exists, and if it does exist, if the solution is unique. For linear differential equations, we have a theorem that's even nicer than our result from section 1.3.

Theorem - If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

has a unique solution $y(x)$ on I.

Note that we're guaranteed a unique solution on the *entire* interval I, not just on some possibly smaller interval like we had for the theorem from section 1.3. Linear differential equations are nice that way.

As a first application of linear first-order equations, we consider a tank containing a solution - a mixture of solute and solvent - such as salt dissolved in water. There is both inflow and outflow, and we want to compute the *amount* $x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the solution in the tank - kept thoroughly mixed by stirring - flows out at the constant rate of r_o liters per second.

The amount of solute flowing into the tank will be

$$r_i c_i,$$

while if c_o is the concentration of the outgoing solution the amount of solute flowing out of the tank will be

$$r_o c_o.$$

So, if $x(t)$ represents the amount of solute in the tank, its rate of change will be:

$$\frac{dx}{dt} = r_i c_i - r_o c_o.$$

Now, we'll usually assume r_i, r_o, and c_i are constant, but the output concentration might very well be changing over time. So, c_o will be given by

$$c_o = \frac{x(t)}{V(t)}.$$

Here $V(t)$ is the volume of water in the tank, which itself might be changing over time. Well, if we plug this in for c_o we get a linear first-order differential equation! Namely,

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{V} x.$$

Example - A 120-gallon (gal) tank initially contains 90 lbs of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at a rate of 4 gal/min, and the well-stirred mixture flows out of the tank at a rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution - The differential equation modeling this system, where $x(t)$ is the amount of salt in the solution at time t, is:

$$\frac{dx}{dt} = (4\ gal/min)(2lb/gal) - \frac{(3\ gal/min)}{90+t}x(t).$$

We can rewrite this as:

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

The integrating factor here will be:

$$\rho = e^{\int \frac{3}{90+t}dt} = e^{3\ln(90+t)} = (90+t)^3.$$

Multiplying both sides by ρ and integrating we get:

$$\int \frac{d}{dt}((90+t)^3 x)dt = \int 8(90+t)^3 dt$$
$$\Rightarrow (90+t)^3 x(t) = 2(90+t)^4 + C.$$

So, the function $x(t)$ will be:

$$x(t) = 2(90+t)^4 + \frac{C}{(90+t)^3}.$$

If we plug in our initial condition we get:

$$x(0) = 2(90 + 0) + \frac{C}{90^3} = 90.$$
$$\Rightarrow C = -(90^4).$$

So, our solution is:

$$x(t) = 2(90 + t) - \frac{90^4}{(90 + t)^3}.$$

If we plug in $t = 30$ we get:

$$x(30) = 2(90 + 30) - \frac{90^4}{(90 + 30)^3} \approx 202 \text{ lbs of salt.}$$