Math 2280 - Lecture 40

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In today's lecture, the final for this course, we'll discuss how Fourier series can be used to solve a simple, but very important *partial* differential equation. Namely, the one-dimensional heat equation.

This lecture corresponds with section 9.5 from the textbook. The assigned (extra credit) problems are:

Section 9.5 - 1, 3, 5, 7, 9

Heat Conduction and Separation of Variables

The flow of heat through a long, thin rod can be modeled by the *one-dimensional heat equation*:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Here, $u(x,t)$ is a function of both displacement, x, and time, t, and k is a given positive constant called the *thermal diffusivity*.

We want to solve this equation for a given set of *boundary conditions*. In ordinary differential equations, the boundary conditions are usually numbers. In partial differential equations, the boundary conditions are usually functions. Here we'll assume our boundary conditions are of the form:

$$u(0,t) = u(L,t) = 0, \ (t > 0),$$
$$u(x,0) = f(x), \ (0 < x < L).$$

The important idea here is that our partial differential equation is *linear*. So, for any two solutions u_1, u_2 we have that $c_1u_1 + c_2u_2$ satisfy the partial differential equation, and if u_1, u_2 satisfy the above boundary conditions on x (called *homogeneous* boundary conditions) then u_1, u_2 will as well. Superposition does *not* work for the boundary condition $u(x,0) = f(x)$, and here is where we need Fourier series. We want to find a linear combination of our almost solutions such that at time $t = 0$ the linear combination is equal to $f(x)$, and gives us a solution.

Example - It is easy to verify by direction substitution that each of the functions:

$$u_1(x,t) = e^{-t}\sin x, \ u_2(x,t) = e^{-4t}\sin 2x, \ u_3(x,t) = e^{-9t}\sin 3x,$$

satisfy the equation $u_t = u_{xx}$. Use these functions to construct a solution to the boundary value problem with boundary values:

$$u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 80\sin^3 x = 60\sin x - 20\sin 3x.$$

Solution - All our functions satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$, and so we want a linear combination such that:

$$c_1e^{-t}\sin x + c_2e^{-4t}\sin 2x + c_3e^{-9t}\sin 3x = 60\sin x - 20\sin 3x$$

when $t = 0$. But this is easy. We can just eyeball it to get $c_1 = 60, c_2 = 0$, and $c_3 = -20$. So, our solution is:

$$u(x,t) = 60e^{-t}\sin x - 20e^{-9t}\sin 3x.$$

That last one was pretty easy. It's also the exception. Usually, we have to find an infinite number of solutions, and make an infinite series equal to $f(x)$. You knew it couldn't be that easy, right?

Separation of Variables

Suppose we have the boundary values $u(x, 0) = u(x, L) = 0$. We're going to assume our function $u(x, t)$ can be written as the product of two functions, one a function of x alone, and the other a function of t alone. This approach is called *separation of variables*. So,

$$u(x, t) = X(x)T(t).$$

Plugging this into our differential equation and doing some algebra we get

$$\frac{X''}{X} = \frac{T'}{kT}.$$

If both X and T are non-trivial functions, this is only possible if both are equal to a constant:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda.$$

This gives us two *ordinary* differential equations. We're now back to familiar territory.

$$X'' + \lambda X = 0,$$
$$T' + \lambda kT = 0.$$

The first must satisfy the boundary conditions $X(0) = X(L) = 0$, and so we have an eigenvalue problem like the ones we dealt with in section 3.8.[1] Well, if we recall section 3.8, we'll remember that the allowable values of λ are

$$\lambda_n = \frac{n^2\pi^2}{L^2},$$

[1] Bet you thought you were done with those, didn't you?

and the eigenfunctions are

$$X_n(x) = \sin\frac{n\pi x}{L}.$$

If we plug this value for λ into our differential equation for T we get:

$$T_n' + \frac{n^2\pi^2 k}{L^2}T_n = 0,$$

A non-trivial solution to this differential equation is:

$$T_n(t) = e^{-n^2\pi^2 kt/L^2}.$$

So, our solution will be:

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.$$

We just need to determine what the coefficients c_n are. This ain't so bad. We want to pick the c_n so that they satisfy

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = f(x).$$

But this is the Fourier sine series for $f(x)$ on the interval $0 < x < L$, and so we have:

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

And we've got our solution! Hooray!

Example - Suppose that a rod of length $L = 50cm$ is immersed in steam until its temperature is $u_0 = 100°C$ throughout. At time $t = 0$, its lateral surface is insulated and its two ends are imbedded in ice at $0°C$. Calculate the rod's temperature at its midpoint after half an hour if it is made of (a) iron ($k = .15$); (b) concrete ($k = .005$).

Solution - The boundary value problem for the rod is given by:

$$u_t = ku_{xx},$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = u_0.$$

Now, we've solved the Fourier series for a square wave a bunch of times, so I'll just cut to the chase and give that the Fourier coefficients are

$$b_{2n+1} = \frac{4u_0}{(2n+1)\pi},$$

for the odd coefficients, and the even coefficients are 0. So, the temperature in the rod will be:

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n} \left(e^{-\frac{n^2\pi^2 k}{L^2}t} \right) \sin\left(\frac{n\pi x}{L}\right).$$

Plugging in $u_0 = 100, L = 50$, and $k = .15$ (for iron) we get that $u(25, 1800) \approx 43.85°C$. Doing the same with $k = .005$ (for concrete) we get $u(25, 1800) \approx 100.00°C$. So, concrete is a *very* good insulator.