Math 2280 - Lecture 34

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Today, we'll develop some of our machinery for using Fourier series, and see how we can use these Fourier series to solve some simple differential equations.

Today's lecture corresponds with section 9.3 from the textbook. The assigned problems for this section are:

Section 9.3 - 1, 5, 8, 13, 20

Fourier Sine and Cosine Series

First, we recall some basic facts about odd and even functions. A function is *odd* if it satisfies:

$$f(-t) = -f(t);$$

while a function is *even* if it satisfies:

$$f(-t) = f(t).$$

If $f(t)$ is an even function, then:

$$\int_{-a}^{a} f(t)dt = 2\int_{0}^{a} f(t)dt;$$

while if $f(t)$ is an odd function:

$$\int_{-a}^{a} f(t)dt = 0.$$

Using just these basic facts we can figure out some important properties of Fourier series for odd or even functions.

The product of two odd or two even functions is even, while the product of an odd and even function is odd. This is analogous to adding integers, which makes sense if you think of an even function as a function whose Taylor series consists of nothing but even terms, and an odd function as a funciton whose Taylor series consists of nothing but odd terms.

So, if $f(t)$ is even then

$$\frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt = 0,$$

and only the a_n terms, a.k.a. the cosine terms, show up in the Fourier series for the function $f(t)$.

Similarly, if $f(t)$ is odd then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = 0,$$

and only the b_n terms survive.

Odd and Even Extensions

Suppose we have a function $f(t)$ defined on the open interval $0 < t < L$. We can define the odd extension of this function on the interval $-L \leq t \leq L$ as:

$$f_{odd}(t) = \begin{cases} f(t) & 0 < t < L \\ -f(-t) & -L < t < 0 \\ 0 & t = \{-L, 0, L\} \end{cases}$$

This function will be odd on the interval $[-L, L]$, and will be equal to $f(t)$ where $f(t)$ is defined. We can then further extend this function to the entire real line by defining it to be $2L$ periodic. This extension to the entire real line is called the *odd extension* of the function $f(t)$. We note that this odd extension is, unsurprisingly, an odd function.

In the same manner we can define an even extension for the function $f(t)$ on the interval $-L \leq t \leq 0$ as:

$$f_{even}(t) = \begin{cases} f(t) & 0 < t < L \\ f(-t) & -L < t < 0 \\ \lim_{t\to 0^+} f(t) & t = 0 \\ \lim_{t\to L^-} f(t) & t = \pm L \end{cases}$$

We note here that we've assumed the limits here are well defined. If they're not, we just say that our even extension is undefined at the corresponding points. As these are isolated points they won't affect integrals, and so won't matter in our calculations of Fourier series. The *even extension* of the function $f(t)$ is just the function defined above made periodic over the whole real line. Again, we note that the even extension is equal to the original function $f(t)$ on the interval upon which $f(t)$ is defined.

So, for any function $f(t)$ defined on the interval $0 < t < L$ we have an odd extension and an even extension, and both these extensions agree with $f(t)$ on the interval $0 < t < L$.

Example - Graph the odd and even extensions of the functions:

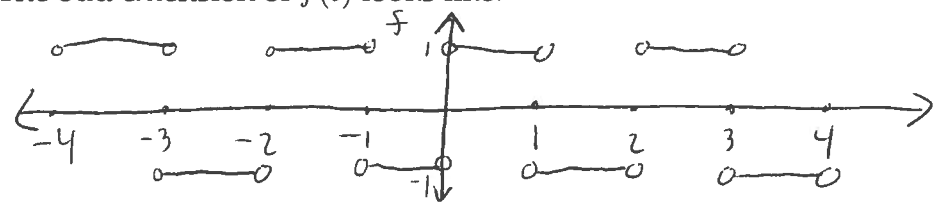
$f(t) = 1$; for $0 < t < 1$,

and

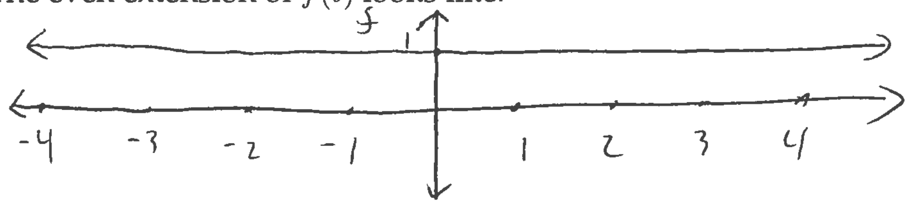
$g(t) = t$; for $0 < t < 1$.

Solution

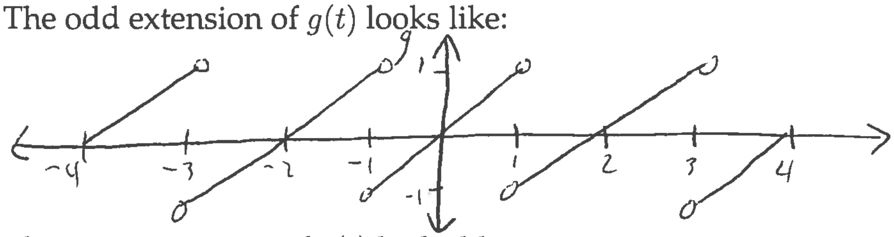
The odd extension of $f(t)$ looks like:



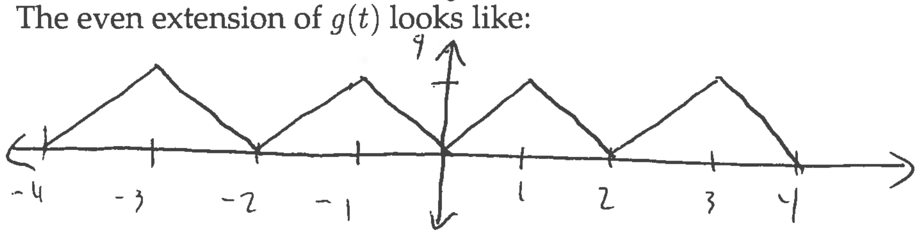
The even extension of $f(t)$ looks like:



The odd extension of $g(t)$ looks like:



The even extension of $g(t)$ looks like:



Now, using our earlier results from odd and even functions, we can see that the Fourier series for the odd extension of the function $f(t)$ will only have sine terms, while the Fourier series for the even extension of the function $f(t)$ will only have cosine terms. These Fourier series are called the Fourier sine and Fourier cosine series, respectively, for the function $f(t)$.

Fourier Series and Differential Equations

This is a differential equations class, so of course what we're going to want to do with Fourier series is use them to solve differential equations. In order to do this, we'll need to know how to differentiate a Fourier series, which is where this next theorem comes into play.

Theorem - Suppose that the function f is continuous and piecewise smooth for all t, and is periodic with period $2L$. Then the Fourier series of $f'(t)$ is the series:

$$f'(t) = \sum_{n=1}^{\infty} \left(-\frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right) + \frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) \right)$$

obtained by termwise differentiation of the Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

Using this result, we can use Fourier series to find solutions to differential equations.

Example - Find a Fourier series solution to the endpoint value problem:

$$x''(t) + 4x(t) = 4t$$
$$x(0) = x(1) = 0.$$

Solution - We want to choose a periodic extension $f(t) = 4t$ such that each term in its Fourier series satisfies the endpoint conditions. For this purpose we want to choose the odd extension. Using the odd extension for the function $f(t)$ we get:

$$b_n = 2\int_0^1 4t \sin(n\pi t)dt = \frac{8}{n^2\pi^2}\left(-(n\pi t)\cos(n\pi t) + \sin(n\pi t)\right)\Big|_0^1$$
$$= \frac{8}{n\pi}(-1)^{n+1}.$$

So, the Fourier series for the odd extension will be:

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi t).$$

So, we'd expect a sine series:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

$$x'(t) = \sum_{n=1}^{\infty}(n\pi)b_n \cos(n\pi t)$$

$$x''(t) = \sum_{n=1}^{\infty} -(n\pi)^2 b_n \sin(n\pi t).$$

So, we have the relation:

$$\sum_{n=1}^{\infty}[-(n\pi)^2+4]b_n \sin(n\pi t) = \frac{8}{\pi}\left(\frac{(-1)^{n+1}}{n}\right)\sin(n\pi t).$$

Equating both sides here we get:

$$b_n = \frac{8\cdot(-1)^{n+1}}{n\pi(4-n^2\pi^2)}.$$

So, we get:

$$x(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi t)}{n(4-n^2\pi^2)}.$$

Now, if we solve this using the method of undetermined coefficients, we'd get that our solution would be:

$$x(t) = t - \frac{\sin(2t)}{\sin 2}, 0 \le t \le 1.$$

As a homework exercise you can calculate that the Fourier series for the odd extnesion of the function $x(t)$ above is indeed the Fourier series we calculated as our solution to the ODE.

Now, in closing we note that we can integrate Fourier series termwise as well, this time with less restrictive conditions on the function $f(t)$.

Theorem - Suppose that f is a piecewise continuous periodic function with period $2L$ and Fourier series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right),$$

which may not converge. Then

$$\int_0^t f(s)ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi} \left(a_n \sin\left(\frac{n\pi t}{L}\right) - b_n \left(\cos\left(\frac{n\pi t}{L}\right) - 1\right)\right),$$

with the series on the right-hand side convergent for all t. Note that the integral series is the result of term-by-term integration of the Fourier series for $f(t)$, but, if $a_0 \neq 0$, it's not a Fourier series because of the linear term $\frac{1}{2}a_0 t$.