Math 2280 - Lecture 33

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In the last lecture I introduced the idea of a Fourier series, and we learned how to calculate a Fourier series for a function with period 2π. Of course, not all periodic functions have period 2π, so you may very well ask what we can do, if anything, with arbitrary periodic functions. In today's lecture we'll show that we can generalize our Fourier series formula from functions with period 2π to functions with arbitrary period. Note this lecture is a little shorter than usual.

Today's lecture corresponds with section 9.2 from the textbook. The assigned problems for this section are:

Section 9.2 - 1, 9, 15, 17, 20

General Fourier Series and Convergence

Suppose we have a function $f(t)$ with period $2L$, where $L > 0$. Then if we define:

$$g(u) = f\left(\frac{Lu}{\pi}\right),$$

we see $g(u)$ is 2π periodic. So, the Fourier series, as defined in the last lecture, for $g(u)$ is:

$$g(u) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nu) + b_n \sin(nu)).$$

The coefficients are:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(u) \cos(nu) du,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(u) \sin(nu) du.$$

We note that $t = \frac{Lu}{\pi}$, and so $f(t) = g\left(\frac{\pi t}{L}\right)$. From this, we get the corresponding "Fourier series"[1] for $f(t)$:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right),$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

So, this generalizes the concept of a Fourier series to functions of period $2L$, where L is any positive number, and not restricted to π. We note that taking the above integrals from $-L$ to L is, given the $2L$-periodicity of the function, completely arbitrary, and we could integrate instead over *any* connected interval of length $2L$.

Convergence of Fourier Series

We've defined the Fourier series of a function, but what does this Fourier series have to do with the original function? What's important about the Fourier series? Well, this question is answered by the next theorem.

[1] In quotations because, really, this isn't so much a derivation as a motivation for a definition.

The Convergence Theorem - Suppose that the periodic function $f(t)$ is piecewise smooth. Then its Fourier series converges to:

1. the value $f(t)$ at each point where f is continuous
2. the value $\frac{1}{2}[f(t^+) + f(t^-)]$ at each point where $f(t)$ is discontinuous.[2]

Example - Find the Fourier series of a square wave function with period 4:

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

Solution - We first note that $f(t)$ is odd, so all the a_n terms in the Fourier series will be zero. The period here is $4 = 2L$, so the b_n Fourier coefficients are:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi t}{2}dt$$

$$\text{(noting } f(t) \text{ is odd, so } f(t)\sin\tfrac{n\pi t}{2} \text{ is even)}$$

$$= \int_{0}^{2} f(t)\sin\frac{n\pi t}{2}dt$$

$$= \int_{0}^{2} \sin\frac{n\pi t}{2}dt = -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^2 = -\frac{2}{n\pi}((-1)^n - 1)$$

$$= \begin{cases} 0 & n\ even \\ \frac{4}{n\pi} & n\ odd \end{cases}$$

So, our Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n\ odd}\frac{\sin\left(\frac{n\pi t}{2}\right)}{n}.$$

[2]Note that $f(t^+)$ means the limit of f as its argument approaches t "from the right", while $f(t^-)$ means the limit of f as its argument approaches t "from the left".

If we plug in $t = 1$ we get:

$$
\begin{gathered}
f(1) = 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\right) \\
= \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots), \\
\text{and so,} \\
\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)
\end{gathered}
$$

which is the famous Leibniz formula for π!