Math 2280 - Lecture 30

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In today's lecture we're going to discuss how to take Laplace transforms of step functions, how these relate to translations, and how the calculation of Laplace transforms can be simplified for periodic functions.

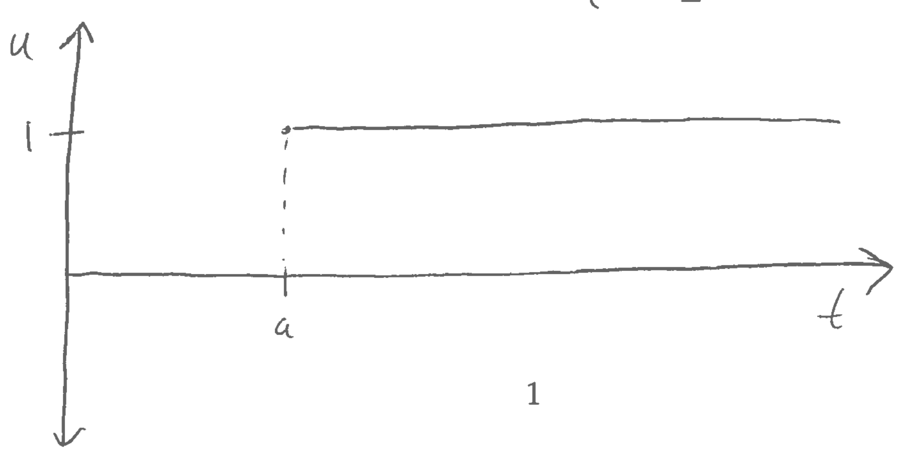
This lecture corresponds with section 7.5 from the textbook. The assigned problems are:

Section 7.5 - 1, 6, 15, 21, 26

Periodic and Piecewise Continuous Input Functions

We begin this lecture by examining a very simple function: the unit step function, which is defined by:

$$u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



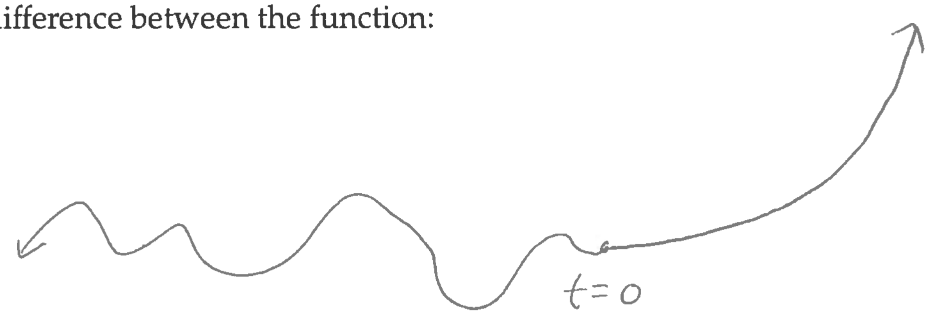
The Laplace transform of this function is:

$$\mathcal{L}(u(t-a)) = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}dt$$
$$= \int_0^\infty e^{-s(t+a)}dt = e^{-as}\int_0^\infty e^{-st}dt = \frac{e^{-as}}{s}$$

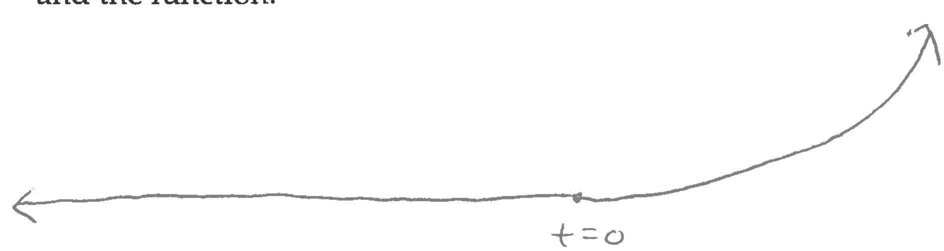
The Laplace transform of $u(t)$ is the same as the Laplace transform of 1, and we see that multiplication of the transform of $u(t)$ by e^{-as} corresponds to the translation $t \to t-a$ of the original independent variable. Turns out this is a general phenomenon.

We'll get into the general theorem in a second, but before going any farther I want to point out that the Laplace transform of the function $f(t)$, and the Laplace transform of the function $u(t)f(t)$ are in fact *the same thing*! This is because when dealing with the Laplace transform we're completely uninterested in what happens for negative values of the function.

You may say this apathy sounds ridiculous. After all, there's a big difference between the function:



and the function:



The point is when we're using Laplace transforms we're usually either assuming that our solution begins at some set time, or we calculate a solution that works for non-negative values, extend this solution to the entire real line, and confirm that it works. In many physical applications this is a very reasonable assumption, because most things "begin", and few things have been going on "forever".

Getting back to step functions, we've seen that in the case of the function $f(t) = u(t)$, multiplying the Laplace transform by e^{-as} is the same as translating the original function by a. This relation holds in general:

Theorem - If $\mathcal{L}(f(t))$ exists for $s > c$, then

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$$

and conversely

$$\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a)$$

for $s > c + a$.

The proof of this I'll leave as an exercise. It follows directly from the definition of the Laplace transform.

Example - Calculate the inverse Laplace transform of:

$$F(s) = \frac{e^{-s} - e^{-3s}}{s^2}$$

Solution

$$\mathcal{L}^{-1}\left(\frac{e^{-s} - e^{-3s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2}\right)$$
$$= u(t-1)(t-1) - u(t-3)(t-3).$$

Example - Calculate the Laplace transform of the function:

$$f(t) = \begin{cases} 2 & 0 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$

Solution - We see that this is the function $f(t) = 2 - 2u(t-3)$, which will have the Laplace transform:

$$\mathcal{L}(f(t)) = \frac{2}{s} - \frac{2e^{-3s}}{s}.$$

Example - Calculate the Laplace transform of the function:

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq 3\pi \\ 0 & t > 3\pi \end{cases}$$

Solution - For $t \geq 0$ this is the function $f(t) = \sin t - u(t-3\pi)\sin t$, which is the same as $f(t) = \sin t + u(t-3\pi)\sin(t-3\pi)$. This second function will have the Laplace transform:

$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + \frac{e^{-3\pi s}}{s^2+1} = \frac{1+e^{-3\pi s}}{s^2+1}.$$

Transforms of Periodic Functions

We say a function defined for $t \geq 0$ is periodic if there is a number $p > 0$ such that

$$f(t+p) = f(t)$$

for all $t \geq 0$. The least positive value of p (if any) for which the equation holds is called *the* period[1] of the function f. If a function is periodic, it's relatively easy to calculate its Laplace transform, and doesn't require the computation of an indefinite integral.

[1]Sometimes the *fundamental period*.

Theorem - Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}(f(t))$ exists for $s > 0$ and is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt.$$

The proof of this theorem is kind of fun, so let's go over it.

Proof - The definition of the Laplace transform gives

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \sum_{n=0}^{\infty} \int_{np}^{(n+1)p} e^{-st} f(t) dt.$$

Now, we note that if we use our periodicity property and the substitution $t = \tau + np$ we get:

$$\int_{np}^{(n+1)p} e^{-st} f(t) dt = \int_0^p e^{-s(\tau + np)} f(\tau + np) d\tau = e^{-nps} \int_0^p e^{-s\tau} f(\tau) d\tau.$$

Using this relation, we see that our Laplace transform is:

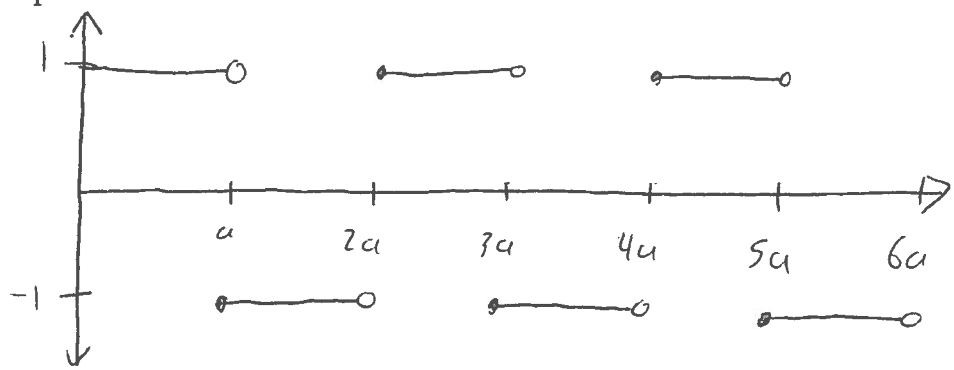
$$F(s) = \sum_{n=0}^{\infty} \left(e^{-nps} \int_0^p e^{-s\tau} f(\tau) d\tau \right) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-s\tau} f(\tau) d\tau.$$

For the final equality we used the geometric series formula:

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1 - x} \qquad\qquad |x| < 1.$$

Example - Calculate the Laplace transform of the square-wave function $f(t) = (-1)^{\lfloor t/a \rfloor}$ of period $p = 2a$.[2]

Graph:



Solution - As mentioned, this is a function with period $2a$, and so applying our formula we get:

$$F(s) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)dt$$

$$F(s) = \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st}dt + \int_a^{2a} (-1)e^{-st}dt \right)$$

$$= \frac{1-2e^{-as}+e^{-2as}}{s(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1}{s} \tanh \frac{as}{2}.$$

[2]The function $\lfloor x \rfloor$ denotes the "greatest integer" function, which means the largest integer not exceding x.

Example - Apply this theorem to verify that $\mathcal{L}(\cos kt) = s/(s^2 + k^2)$.

Solution - This function is periodic with period $2\pi/k$, and so the Laplace transform will be the integral:

$$\mathcal{L}(\cos kt) = \frac{1}{1 - e^{-2\pi s/k}} \int_0^{2\pi/k} e^{-st} \cos{(kt)} dt.$$

Now, if we use the relation:

$$\int e^{-st} \cos{(kt)} dt = \frac{e^{-st}}{k^2 + s^2}(k \sin kt - s \cos kt).$$

we get the solution:

$$\begin{aligned} \mathcal{L}(\cos kt) &= \frac{1}{1 - e^{-2\pi s/k}} e^{-st} \left(\frac{k \sin kt - s \cos kt}{k^2 + s^2} \right) \Bigg|_0^{2\pi/k} \\ &= \frac{s}{k^2 + s^2} \frac{1 - e^{-2\pi s/k}}{1 - e^{-2\pi s/k}} = \frac{s}{k^2 + s^2}. \end{aligned}$$

This is what we wanted. Score!