Math 2280 - Lecture 29

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A few lectures ago we learned that the Laplace transform is linear, which can enormously simplify the calculation of Laplace transforms for sums and scalar multiples of functions. The next natural question is what relations, if any, are there for Laplace transforms of products? It might be "nice" if

$$\mathcal{L}(f(t) \cdot g(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))^{1}$$

But we can see right away that this would be ridiculous:

$$\frac{1}{s} = \mathcal{L}(1) = \mathcal{L}(1 \cdot 1) = \mathcal{L}(1) \cdot \mathcal{L}(1) = \frac{1}{s^2}.$$

That ain't right. So, we must state that, in general

$$\mathcal{L}(f(t) \cdot g(t)) \neq \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)).$$

So, that doesn't work, and the question remains. Namely, what, if any, relation is there between products and Laplace transforms? It turns out there is an operation called the *convolution* of two functions, and this operation will give us the best answer to our question.

[1] This is NOT true. Not true, not true, not true. Don't memorize this because it's NOT TRUE.

In additional to convolutions, today we'll also discuss derivatives and integrals of Laplace transforms, and how they relate to inverse Laplace transforms.

Today's lecture corresponds with section 7.4 of the textbook, and the assigned problems are:

Section 7.4 - 1, 5, 10, 19, 31

Derivatives, Integrals, and Products of Transforms

To answer our question about products and Laplace transforms we first need to define an operator called *convolution*. The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$, and is defined by:

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

We note that it's a straightforward if tedious task to verify that the operator is commutative and associative.

Example - Calculate the convolution:

$$t^2 * \cos t.$$

Solution - The convolution is:

$$\int_0^t \tau^2 \cos(t-\tau)d\tau.$$

Using the relation:

$$\cos(t-\tau) = \cos t \cos\tau + \sin t \sin\tau$$

we get that this is equal to:

$$\cos t \int_0^t \tau^2 \cos \tau d\tau + \sin t \int_0^t \tau^2 \sin \tau.$$

If we use a table of integrals[2] to calculate the integrals above we get:

$$\int \tau^2 \cos \tau d\tau = \tau^2 \sin \tau + 2\tau \cos \tau - 2 \sin \tau,$$

and

$$\int \tau^2 \sin \tau d\tau = -\tau^2 \cos \tau + 2\tau \sin \tau + 2 \cos \tau.$$

And so, our integrals are:

$$\cos t \int_0^t \tau^2 \cos \tau d\tau = t^2 \sin t \cos t + 2t \cos^2 t - 2 \sin t \cos t,$$

and

$$\sin t \int_0^t \tau^2 \sin \tau d\tau = -t^2 \sin t \cos t + 2t \sin^2 t + 2 \sin t \cos t - 2 \sin t.$$

Therefore, the convolution is the sum of the two functions above:

$$2t - 2 \sin t.$$

Now, why are these convolutions important? Well, as you might expect, they relate to products of Laplace transforms in a big way. This relation is given by the next theorem.

[2] Or integrals.com, or Wolfram Alpha, or Mathematica, or Matlab, or Maple, or Eugene the Magical Monkey who does Integrals...

Theorem - Suppose that $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$ and that $|f(t)|$ and $|g(t)|$ are bounded by Me^{ct} as $t \rightarrow \infty$. Then the Laplace transform of $f * g$ exists for $s > c$, and

$$\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t)).$$

Example - Calculate the inverse Laplace transform of:

$$F(s) = \frac{1}{s(s^2+1)}.$$

Solution - We have that:

$$F(s) = \frac{1}{s(s^2+1)} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+1}\right) = \mathcal{L}(1) \cdot \mathcal{L}(\sin t).$$

So, the inverse Laplace transform is:

$$f(t) = 1 * \sin t = \int_0^t \sin \tau d\tau = 1 - \cos t.$$

Finally, just as there are relations for the Laplace transform of a derivative, there are relations for the derivative of a Laplace transform. These are:

$$\mathcal{L}(t \cdot f(t)) = -F'(s),$$

and in greater generality

$$\mathcal{L}(t^n \cdot f(t)) = (-1)^n F^{(n)}(s).$$

Also, conversely

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma) d\sigma.$$

Deriving these relations is fairly straightforward, and is done at the end of section 7.4 from the textbook.

Example - Calculate the Laplace transform:

$$\mathcal{L}(t\sin(3t)).$$

Solution - Using our relation we get that this is equal to:

$$-F'(s),$$

where

$$F(s) = \mathcal{L}(\sin(3t)) = \frac{3}{s^2+9}.$$

Taking the derivative and multiplying by -1 we get:

$$\mathcal{L}(t\sin(3t)) = \frac{6s}{(s^2+9)^2}.$$

Example - Calculate the Laplace transform:

$$\mathcal{L}\left(\frac{e^{3t}-1}{t}\right).$$

Solution - We know the Laplace transform:

$$\mathcal{L}(e^{3t}-1) = \frac{1}{s-3} - \frac{1}{s}.$$

So,

$$\mathcal{L}\left(\frac{e^{3t}-1}{t}\right) = \int_s^\infty \left(\frac{1}{\sigma-3} - \frac{1}{\sigma}\right) d\sigma.$$

Calculating this integral we get:

$$\int_s^{\infty} \left(\frac{1}{\sigma - 3} - \frac{1}{\sigma} \right) d\sigma = \ln(\sigma - 3) - \ln \sigma |_s^{\infty}$$

$$= \ln \left(\frac{\sigma - 3}{\sigma} \right) |_s^{\infty} = -\ln \left(\frac{s-3}{s} \right) = \ln \left(\frac{s}{s-3} \right) \text{ for } s > 3.$$