Math 2280 - Lecture 27

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In our last lecture I introduced the Laplace transform, and we discussed a few of its properties. All nice and good, you may be thinking, but what does it have to do with solving differential equations? I'm so glad you asked. Today, we'll learn about a few more properties of Laplace transforms, and how these properties can be used in figuring out solutions to differential equations.

This lecture corresponds with section 7.2 of the textbook. The assigned problems from this section are:

Section 7.2 - 1, 4, 15, 20, 29

Transformation of Initial Value Problems

Laplace transforms are going to allow us to take differential equations and turn them into algebraic equations. We can then solve these algebraic equations to find solutions to our differential equations. It's pretty slick.

But, before we get into this, we need to establish one *very* important property of the Laplace transform.

Theorem - Suppose that the function $f(t)$ is continuous and piecewise smooth (which means smooth except at finite isolated points) for $t \geq 0$ and is of exponential order as $t \rightarrow \infty$. Then $\mathcal{L}(f'(t))$ exists for some $s > c$, and

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) = sF(s) - f(0), \qquad s > c.$$

From this it follows by induction that:

$$\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{n-2}(0) - f^{n-1}(0),$$
$$s > c.$$

This is huge! What it means is that not only can we take Laplace transforms of functions, we can take Laplace transforms of linear differential equations![1]

Example - Use Laplace transforms to solve the initial value problem:

$$x'' + 9x = 0; x(0) = 3; x'(0) = 4.$$

Solution - If we take the Laplace transform of the differential equation we get:

$$\mathcal{L}(x'' + 9x) = s^2X(s) - sx(0) - x'(0) + 9X(s) = 0.$$

If we use our initial conditions $x(0) = 3$ and $x'(0) = 4$ to solve for $X(s)$ we get:

$$X(s) = \frac{3s+4}{s^2+9} = 3\frac{s}{s^2+9} + 4\frac{1}{s^2+9}.$$

If we look at our table of Laplace transforms we find that, for $s > 0$, this is the Laplace transform of the function:

$$x(t) = 3\cos 3t + \frac{4}{3}\sin 3t.$$

Which is the solution to our initial value problem!

[1]Great Scott!

Let's see that again.

Example - Find the solution to the initial value problem below using Laplace transforms:

$$x'' + 8x' + 15x = 0; x(0) = 2; x'(0) = 3$$

Solution - If we take the Laplace transform of this relation we get:

$$s^2X(s) - sx(0) - x'(0) + 8sX(s) - 8x(0) + 15X(s) = 0.$$

If we plug in our initial conditions and solve for $X(s)$ we get:

$$X(s) = \frac{2s + 19}{s^2 + 8s + 15}.$$

Now, if we do a partial fraction decomposition, noting that the denominator factors as $(s+5)(s+3)$, we get:

$$X(s) = \frac{-\frac{9}{2}}{s+5} + \frac{\frac{13}{2}}{s+3}.$$

In this form the inverse Laplace transform becomes obvious:

$$x(t) = -\frac{9}{2}e^{-5t} + \frac{13}{2}e^{-3t}.$$

Now, just as with differentiation, we have a relation between the Laplace transform of a function and the Laplace transform of the integral of the function.

Theorem - If $f(t)$ is a piecewise continuous function for $t \geq 0$ and satisfies the condition of exponential order, then

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}(f(t)) = \frac{F(s)}{s}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)d\tau.$$

Now, these differentiation and integration rules can be exploited to make the calculation of some Laplace transforms much easier.

Example - Find $\mathcal{L}(t \sin kt)$.

Solution - This becomes much easier if we first differentiate:

$$f'(t) = \sin kt + kt \cos kt$$

and note that $f(0) = f'(0) = 0$. If we differentiate again we get:

$$f''(t) = 2k \cos kt - k^2 t \sin kt.$$

The laplace transform of $f''(t)$ is $s^2 F(s)$, and from this we get the relation:

$$s^2 F(s) = \frac{2ks}{s^2 + k^2} - k^2 F(s).$$

Solving this for $F(s)$ we get:

$$\mathcal{L}(t \sin kt) = F(s) = \frac{2ks}{(s^2 + k^2)^2}.$$

So, we have our solution. We note that this is much easier than actually evaluating the integral:

$$\mathcal{L}(t \sin kt) = \int_0^\infty t e^{-st} \sin kt dt.$$