Math 2280 - Lecture 26

Dylan Zwick

Summer 2013

Today we're going to transition from the study of linear dynamical systems and return to the study of (primarily linear) ordinary differential equations.

In particular, today we'll begin talking about Laplace transforms. These are the first step into a vast sea of transform methods that have dramatically expanded mathematics and continue to find new and unexpected applications.

The section of the textbook corresponding to today's lecture is section 7.1, and the assigned problems are:

Section 7.1 - 1, 6, 20, 30, 36

Laplace Transforms and Inverse Problems

A Laplace transformation is a map from functions to functions. You've already seen many such maps, even if you didn't realize it at the time those maps were introduced. Differentiation, for example, is a map from functions to functions. The differential operator would map the function $f(t) = t^2$ to:

$$D_t(f(t)) = 2t = f'(t).$$

We have an operator, the differential operator, that takes a function $f(t)$ as its input, and outputs another function, denoted $f'(t)$. This operator is not well defined for all input functions. In fact, it's only defined on a particular class of functions called, not surprisingly, the differentiable functions.

The Laplace transform is another map from functions to functions. Just as not every function is differentiable, not every function has a well-defined Laplace transform. There's a certain set of functions that are in the domain (allowable inputs) of the Laplace transform operator. Furthermore, the domain of the output function frequently is not all real numbers, even which the domain of the input function is. So, when giving the output of a Laplace transform, it's important to specify the domain upon which this output is defined.

Taking the function $f(t) = t^2$ from our earlier example, the Laplace transform of this input would be:

$$\mathcal{L}(t^2) = \frac{2}{s^3}, \qquad s > 0.$$

The domain of $\mathcal{L}(t^2)$ is all positive numbers, not all real numbers. To see why this is so, and how we calculated this, we of course need to define the Laplace transform. This is what we'll do next.

Definitions and Properties

Definition - Given a function $f(t)$ defined for all $t \geq 0$, the *Laplace transform* of f is the function F defined by:

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$$

for all values of s where the improper integral converges.

We recall that what we mean by a limit of integration being infinity is that it's defined in terms of the limit:

$$\int_0^\infty g(t)dt = \lim_{b\to\infty}\int_0^b g(t)dt$$

and this only makes sense if the limit is well defined.

Let's use this definition to actually calculate the Laplace transform stated above.

Example - Calculate the Laplace transform of $f(t) = t^2$.

Solution -

$$\begin{aligned}
\mathcal{L}(t^2) &= \int_0^\infty e^{-st}t^2 dt \\
&= -\frac{e^{-st}t^2}{s}\bigg|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}t dt \\
&= -\frac{2}{s^2}te^{-st}\bigg|_0^\infty + \frac{2}{s^2}\int_0^\infty e^{-st}dt \\
&= -\frac{2}{s^3}e^{-st}\bigg|_0^\infty = \frac{2}{s^3}, \qquad s > 0.
\end{aligned}$$

Note that we had to assume $s > 0$ in order for the integral to converse.

Now, just based upon the definition and the linearity of integration we can deduce that the Laplace transform, like differentiation, is a linear operator:

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

where $f(t)$ and $g(t)$ are functions of t and a, b are constants.

Example - Calculate the Laplace transform of $4t^2$.

Solution -

$$\mathcal{L}(4t^2) = 4\mathcal{L}(t^2) = \frac{8}{s^3}, \qquad s > 0.$$

Some Common Laplace Transforms

Let's use the definition of the Laplace transform to calculate the Laplace transforms of some common functions.

Example - Calculate the Laplace transform of $f(t) = 1$.

Solution -

$$\mathcal{L}(1) = \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_0^\infty = \frac{1}{s}, \qquad s > 0.$$

Example - Calculate the Laplace transform of $f(t) = 3t^2 + 5$.

Solution -

$$\mathcal{L}(3t^2 + 5) = 3\mathcal{L}(t^2) + 5\mathcal{L}(1) = \frac{6}{s^3} + \frac{5}{s}, \qquad s > 0.$$

That last calculation was made pretty easy using the linearity property. This is a fact we'll want to use frequently. Again as with differentiation, calculating a Laplace transform using the formal definition can be a major pain. Whenever it can be avoided, it should be.

Let's generalize a bit and talk about how to take Laplace transforms of functions of the form t^a, where a is a real number and $a > -1$. Well, to do this we're going to first want to define a very interesting function called the *gamma function*, which is defined for $x > 0$ by:

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt.$$

Now, it's the matter of some simple integration to check that:

$$\Gamma(1) = 1,$$

and using integration by parts we get the relation:

$$\Gamma(x+1) = x\Gamma(x).$$

This is interesting. If we say that x is an integer, n, we get:

$$\Gamma(n+1) = n!$$

I'd like to put an exclamation mark at the end of that sentence, but there's already one there. In fact, this is how we technically want to *define* the factorial, and it gives an explanation as to why $0! = 1$ that is more than just ad hoc. So, to recap, the $Gamma$ function is a function that is defined and continuous (we won't prove continuity, but trust me) for $x > -1$, and is equal to $(n-1)!$ when x is any natural number.

Now, what does the Gamma function have to do with Laplace transforms? I'm glad you asked. If we want to take the Laplace transform of a function of the form t^a with $a > -1$ we want to calculate the integral:

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a dt.$$

If we make the substitution $u = st$ we get:

$$\mathcal{L}(t^a) = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{\Gamma(a+1)}{s^{a+1}}, \qquad s > 0$$

In particular, if a is a natural number n we get:

$$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}, \qquad s > 0.$$

which is consistent with our results for t^2 and 1.

Using the definition of the Laplace transform and some calculus we can calculate the following relations, some of which are done for you in the textbook. The textbook also has a useful table of Laplace transforms on page 446.

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \qquad s > a$$
$$\mathcal{L}(\cos kt) = \frac{s}{s^2+k^2}, \qquad s > 0$$
$$\mathcal{L}(\sin kt) = \frac{k}{s^2+k^2}, \qquad s > 0$$
$$\mathcal{L}(\cosh kt) = \frac{s}{s^2-k^2}, \qquad s > 0$$
$$\mathcal{L}(\sinh kt) = \frac{k}{s^2-k^2}, \qquad s > 0$$

Example - What is the Laplace transform of $t - 2e^{3t}$?

Solution -

$$\mathcal{L}(t - 2e^{3t}) = \mathcal{L}(t) - 2\mathcal{L}(e^{3t}) = \frac{1}{s^2} - \frac{2}{s-3}, \qquad s > 3.$$

Example - What is the Laplace transform of $\sin 3t + \cos 2t$?

Solution -

$$\mathcal{L}(\sin 3t + \cos 2t) = \frac{3}{s^2+9} + \frac{s}{s^2+4}, \qquad s > 0.$$

Important Properties of the Laplace Transform

We'll state without proof some important properties of the Laplace transform. If you're interested in the proofs they're either in your textbook, or in references provided by your textbook.

First, we say that a function $f(t)$ is of *exponential order* as $t \to \infty$ if there exist nonnegative constants $M, c,$ and T such that:

$$|f(t)| \leq Me^{ct} \text{ for } t \geq T.$$

In other words, eventually it's bounded by some exponential function with a linear argument.

So, for example, every polynomial is of exponential order, while the function e^{t^2} isn't.

Theorem - If the function f is piecewise continuous for $t \geq 0$ and is of exponential order as $t \to \infty$, then its Laplace transform $F(s) = \mathcal{L}(f(t))$ exists. More precisely, if f is piecewise continuous and satisfies the condition defined above, then $F(s)$ exists for all $s > c$.

Corollary - A result you may have guessed from our examples so far, and which comes out from the proof of the theorem above, is that when the function $f(t)$ satisfies the requirements in our theorem then:

$$\lim_{s\to\infty} F(s) = 0.$$

Finally, we state a *very* important theorem, especially for what we're going to be doing with Laplace transforms.

Theorem - Suppose that the functions $f(t)$ and $g(t)$ are of exponential order and piecewise continuous, so their Laplace transforms both exist. If $F(s) = G(s)$ for all $s > c$ (for some c), then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.

This property is extremely important, because what it says is that if we're given a Laplace transform $F(s)$ we can make sense of what we mean by the inverse transform. That is, the "unique" (except for isolated points) function $f(t)$ whose Laplace transform is $F(s)$.

Example - Find the inverse Laplace transform of $F(s) = \frac{1}{s+5}$.

Solution -

$$\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}.$$

Example - Find the inverse Laplace transform of $F(s) = \frac{3s+1}{s^2+4}$

Solution -

$$\mathcal{L}^{-1}\left(\frac{3s+1}{s^2+4}\right) = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = 3\cos 2t + \frac{1}{2}\sin 2t.$$