Math 2280 - Lecture 23

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In our last lecture we dealt with solutions to the system:

$$\mathbf{x}' = A\mathbf{x}$$

where A is an $n \times n$ matrix with n distinct eigenvalues. As promised, today we will deal with the question of what happens if we have less than n distinct eigenvalues, which is what happens if any of the roots of the characteristic polynomial are repeated.

This lecture corresponds with section 5.4 of the textbook, and the assigned problems from that section are:

Section 5.4 - 1, 8, 15, 25, 33

The Case of an Order 2 Root

Let's start with the case an an order 2 root.[1] So, our eigenvalue equation has a repeated root, λ, of multiplicity 2.

There are two ways this can go. The first possibility is that we have two distinct (linearly independent) eigenvectors associated with the eigenvalue λ. In this case, all is good, and we just use these two eigenvectors to create two distinct solutions.

[1] Admittedly, not one of Sherlock Holmes's more popular mysteries.

Example - Find a general solution to the system:

$$\mathbf{x}' = \begin{pmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{pmatrix} \mathbf{x}$$

Solution - The characteristic equation of the matrix $\mathbf{A}$ is:

$$|\mathbf{A} - \lambda \mathbf{I}| = (5 - \lambda)(3 - \lambda)^2.$$

So, $\mathbf{A}$ has the distinct eigenvalue $\lambda_1 = 5$ and the repeated eigenvalue $\lambda_2 = 3$ of multiplicity 2.

For the eigenvalue $\lambda_1 = 5$ the eigenvector equation is:

$$(\mathbf{A} - 5\mathbf{I})\mathbf{v} = \begin{pmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which has as an eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Now, as for the eigenvalue $\lambda_2 = 3$ we have the eigenvector equation:

$$\begin{pmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For this eigenvector equation we have two linearly independent eigenvectors:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}.$$

So, we have a complete set of linearly independent eigenvectors, and associated solution:

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{5t} + c_2\mathbf{v}_2 e^{3t} + c_3\mathbf{v}_3 e^{3t}$$

Well, that was no problem. Unfortunately, as we will see momentarily, it isn't always the case that we can find two linearly independent eigenvectors for the same eigenvalue.

Example - Calculate the eigenvalues and eigenvectors for the matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$$

Solution - We have characteristic equation $(\lambda - 4)^2 = 0$, and so we have a root of order 2 at $\lambda = 4$. The corresponding eigenvector equation is:

$$(\mathbf{A} - 4\mathbf{I}) = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We get one eigenvector:

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and that's it!

In this situation we call this eigenvalue *defective*, and the *defect* of this eigenvalue is the difference beween the multiplicity of the root and the number of linearly independent eigenvectors. In the example above the defect is of order 1.

Now, how does this relate to systems of differential equations, and how do we deal with it if a defective eigenvalue shows up?[2] Well, suppose we have the same matrix $\mathbf{A}$ as above, and we're given the differential equation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

For this differential equation we know how to find one solution:

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$$

but for a complete set of solutions we'll need another linearly independent solution. How do we get this second solution?

Based on experience we may think that, if $\mathbf{v}_1 e^{\lambda t}$ is a solution, then we can get a second solution of the form $\mathbf{v}_2 t e^{\lambda t}$. Let's try out this solution:

$$\mathbf{v}_2 e^{\lambda t} + \lambda \mathbf{v}_2 t e^{\lambda t} = \mathbf{A}\mathbf{v}_2 t e^{\lambda t}.$$

Unfortunately for this to be true it must be true when $t = 0$, which would imply that $\mathbf{v}_2 = \mathbf{0}$, which wouldn't be a linearly independent solution. So, this approach doesn't work.[3]

But wait, there's hope. We don't have to abandon this method entirely.[4] Suppose instead we modify it slightly and replace $\mathbf{v}_2 t$ with $\mathbf{v}_1 t + \mathbf{v}_2$, where $\mathbf{v}_1$ is the one eigenvector that we were able to find. Well, if we plug this into our differential equation we get the relations:

$$\mathbf{v}_1 e^{\lambda t} + \lambda \mathbf{v}_1 t e^{\lambda t} + \lambda \mathbf{v}_2 e^{\lambda t} = \mathbf{A}\mathbf{v}_1 t e^{\lambda t} + \mathbf{A}\mathbf{v}_2 e^{\lambda t}.$$

[2]No, run and hide is not an acceptable answer.
[3]Nuts!
[4]Hooray!

If we equate the $e^{\lambda t}$ terms and the $te^{\lambda t}$ terms we get the two equalities:

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = 0$$

and

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1.$$

Let's think about this. What this means is that, if we have a defective eigenvalue with defect 1, we can find two linearly independent solutions by simply finding a solution to the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2\mathbf{v}_2 = \mathbf{0}$$

such that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 \neq \mathbf{0}.$$

If we can find such a vector $\mathbf{v}_2$, then we can construct two linearly independent solutions:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}$$

where $\mathbf{v}_1$ is the non-zero vector given by $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2$.

For our particular situation we have:

$$(\mathbf{A} - 4\mathbf{I})^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, *any* non-zero vector could potentially work as our vector $\mathbf{v}_2$. If we try

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then we get:

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

So, our linearly independent solutions are:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{4t} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t},$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{4t} = \begin{pmatrix} -3t + 1 \\ 3t \end{pmatrix} e^{4t}.$$

with corresponding general solution:

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t).$$

We note that our eigenvector $\mathbf{v}_1$ is not our original eigenvector, but is a multiple of it. That's fine.

The General Case

The vector $\mathbf{v}_2$ above is an example of something called a *generalized eigenvector*. If λ is an eigenvalue of the matrix $\mathbf{A}$, then a *rank r generalized eigenvector* associated with λ is a vector $\mathbf{v}$ such that:

$$(\mathbf{A} - \lambda \mathbf{I})^r \mathbf{v} = \mathbf{0}$$

but

$$(\mathbf{A} - \lambda \mathbf{I})^{r-1} \mathbf{v} \neq \mathbf{0}.$$

We note that a rank 1 generalized eigenvector is just our standard eigenvector, where we treat a matrix raised to the power 0 as the identity matrix.

We define a *length k chain of generalized eigenvectors based on the eigenvector $\boldsymbol{v}_1$* as a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of k generalized eigenvectors such that

$$
\begin{aligned}
(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_k &= \mathbf{v}_{k-1} \\
(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_{k-1} &= \mathbf{v}_{k-2} \\
&\vdots \\
(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1.
\end{aligned}
$$

Now, a fundamental theorem from linear algebra states[5] that every $n \times n$ matrix $\mathbf{A}$ has n linearly independent generalized eigenvectors. These n generalized eigenvectors may be arranged in chains, with the sum of the lengths of the chains associated with a given eigenvalue λ equal to the multiplicity of λ (a.k.a. the order of the root in the characteristic equation). However, the structure of these chains can be quite complicated.

The idea behind how we build our solutions is that we calculate the defect d of our eigenvalue λ and we figure out a solution to the equation:

$$(\mathbf{A} - \lambda \mathbf{I})^{d+1}\mathbf{u} = \mathbf{0}.$$

We then successively multiply by the matrix $(\mathbf{A} - \lambda \mathbf{I})$ until the zero vector is obtained. The sequence gives us a chain of generalized eigenvectors, and from these we build up solutions as follows (assuming there are k generalized eigenvectors in the chain):

$$
\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{v}_1 e^{\lambda t} \\
\mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2)\, e^{\lambda t} \\
\mathbf{x}_3(t) &= \left(\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3\right) e^{\lambda t}
\end{aligned}
$$

[5]This is mathspeak for "A theorem that's true but we're not going to prove. So just trust me."

$$\vdots$$

$$\mathbf{x}_k(t) = \left(\mathbf{v}_1 \frac{t^{k-1}}{(k-1)1} + \cdots + \mathbf{v}_{k-2}\frac{t^2}{2!} + \mathbf{v}_{k-1}t + \mathbf{v}_k\right)e^{\lambda t}$$

We then amalgamate all these chains of generalized eigenvectors, and these gives us our complete set of linearly independent solutions.[6] This always works. We note that in all the examples we're doing we're assuming all our eigenvalues are real, but that assumption isn't necessary. This method works just fine if we have complex eigenvalues, as long as we allow for complex eigenvectors as well.

Example - Find the general solution of the system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix}\mathbf{x}.$$

Solution - The characteristic equation of the coefficient system is:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1 & 2 \\ -5 & -3-\lambda & -7 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda+1)^3$$

and so the matrix $\mathbf{A}$ has the eigenvalue $\lambda = -1$ with multiplicity 3. The corresponding eigenvector equation is:

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The only eigenvectors that work here are of the form:

[6] In practice we'll only be dealing with smaller (2x2, 3x3, maybe a 4x4) systems, so things won't get too awful.

$$\begin{pmatrix} a \\ a \\ -a \end{pmatrix}$$

and so the eigenvalue $\lambda = -1$ has defect 2. In order to figure out the generalized eigenvectors, we need to calculate $(\mathbf{A} - \lambda\mathbf{I})^2$ and $(\mathbf{A} - \lambda\mathbf{I})^3$:

$$(\mathbf{A} - \lambda\mathbf{I})^2 = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$(\mathbf{A} - \lambda\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, what we want to do is find a vector $\mathbf{v}$ such that $(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0}$ (not hard to do) but also such that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} \neq \mathbf{0}$ (also not hard to do, but perhaps not trivial). Let's try the simplest vector we can think of:

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

If we test this out we get:

$$(\mathbf{A} - \lambda\mathbf{I})^3\mathbf{v} = \mathbf{0} \text{ (obviously)}$$
$$(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{v} = \begin{pmatrix} -2 & -1 & -3 \\ -2 & -1 & -3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, all is good here. We say $\mathbf{v} = \mathbf{v}_3$, and use it to get the other vectors in our chain:

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = \mathbf{v}_2$$

$$\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \mathbf{v}_1$$

Using these three generalized eigenvectors we recover our three linearly independent solutions:

$$\mathbf{x}_1(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} e^{-t}$$

$$\mathbf{x}_2(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} e^{-t}$$

$$\mathbf{x}_3(t) = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \frac{1}{2}t^2 e^{-t} + \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t}$$

and so our general solution will be:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3$$

where the constants c_1, c_2 and c_3 are determined by initial conditions.