Math 2280 - Lecture 22

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So far we've examined some systems of first-order differential equations, and we've learned how to solve those systems by using the method of elimination. Using this method, we reduced the system to a single higher-order differential equation that we then know how to solve using the methods from chapter 3. We've also seen how to rewrite a higher-order differential equation as a system of first-order differential equations. Wouldn't it be nice if there were a way to solve a system of first-order differential equations without converting the system into a higher-order differential equation?

Well, for once, the universe is nice. There *is* a method for solving systems of first-order differential equations, and it involves finding those eigenvalues you got to know and love in math 2270. Today we'll learn about this method.

Today's lecture corresponds with section 5.2 from the textbook. The assigned problems from this section are:

Section 5.2 - 1, 9, 15, 21, 39

The Eigenvalue Method for Homogeneous Systems

Suppose we have a system of first-order ODEs with *constant* coefficients:

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\begin{aligned} x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

We know that *any* solution (general theory) can be written as the linear combination:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

where the $\mathbf{x}_i$ are linearly independent solutions of the system of ODEs. So, what we want to do is figure out how to find these linearly independent solutions.

The Exponential "Guess"

By analogy with the constant coefficient case for homogeneous ODEs, we can "guess" a solution of the form:

$$\mathbf{x}(t) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} e^{\lambda t} = \mathbf{v}e^{\lambda t}$$

where the v_i and λ are appropriate scalar constants.

Now, if we write our system as:

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

then if $\mathbf{x} = \mathbf{v}e^{\lambda t}$ we get:

$$\lambda\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t}$$
$$\Rightarrow \lambda\mathbf{v} = \mathbf{A}\mathbf{v}$$

which is the "eigenvalue equation" from linear algebra.

The Eigenvalue Equation

We begin with a theorem from linear algebra. Namely, that $\mathbf{Av} = \lambda\mathbf{v}$ for some $\mathbf{v} \neq 0$ if and only if $det(\mathbf{A} - \lambda\mathbf{I}) = 0$. This theorem determines the possible values of λ.

In general

$$det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

gives us a "characteristic"[1] nth-order (in λ) polynomial whose roots are the acceptable values of λ.

Well, if we get n *distinct* eigenvalues, as these roots are called, then we get n linearly independent solutions, and we're done. Now, as you might imagine, these solutions may be complex conjugates, a situation we'll discuss today. We'll delay what we do if any of the eigenvalues are repeated until next time.

All Real Roots

If all the roots are real and distinct, then the problem is as easy as it can be. How this is handled is best seen in the context of an example.

Example - Find the general solution of:

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x}$$

Solution - First, we find the eigenvalues of the matrix

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}.$$

[1]The German term "eigen" roughly translates, in this context, as "characteristic".

To do this, we find the roots of the characteristic equation:

$$\left| \begin{array}{cc} 2-\lambda & 3 \\ 2 & 1-\lambda \end{array} \right| = (2-\lambda)(1-\lambda) - (3)(2) = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1).$$

So, the roots are $\lambda = 4$ and $\lambda = -1$, which are our eigenvalues.

Next, we find corresponding eigenvectors:

For $\lambda = 4$:

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we get, for example,

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$ [2]

For $\lambda = -1$:

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we get, for example,

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So, the general solution is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

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[2]Note that any non-zero constant multiple of $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ would also work here as an eigenvector.

Complex Eigenvalues

Any complex eigenvalue will also have its conjugate as an eigenvalue:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$$
$$\Rightarrow (\mathbf{A} - \overline{\lambda}\mathbf{I})\overline{\mathbf{v}} = 0$$

So, $\overline{\mathbf{v}}$ is a corresponding eigenvector to the eigenvalue $\overline{\lambda}$. Now, if λ is complex then we have:

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} = \mathbf{v}e^{(p+qi)t} = (\mathbf{a} + \mathbf{b}i)e^{pt}(\cos(qt) + i\sin(qt))$$

which gives us,

$$\mathbf{x}(t) = e^{pt}(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)) + ie^{pt}(\mathbf{b}\cos(qt) + \mathbf{a}\sin(qt))$$

Now, as $0 = 0 + i0$, both the real term and complex term here must be a solution to the system of ODEs, and these are the same pair of solutions we'll get from the eigenvalue's conjugate. So, our two linearly independent solutions, arising from the eigenvalue and its conjugate, are the real and imaginary parts above.

Example - Find the solution to the given system of ODEs:

$$x_1' = x_1 - 2x_2$$
$$x_2' = 2x_1 + x_2$$
$$x_1(0) = 0, x_2(0) = 4$$

Solution - Again, the first thing we do is find the eigenvalues of the matrix:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

This involves finding the roots of the characteristic equation:

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (2)(-2) = \lambda^2 - 2\lambda + 5.$$

Using the quadratic equation we get:

$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

So, our eigenvalues are the complex conjugate pair $1 \pm 2i$. The eigenvector corresponding to $\lambda = 1 - 2i$ will be:

$$\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

From which we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvector corresponding to $\lambda = 1 + 2i$ will just be the conjugate of the eigenvector above. So, the general solution to our system will be:

$$\mathbf{x}(t) =$$
$$c_1 e^t \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t \right) + c_2 e^t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t \right).$$

From our initial conditions we get:

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which means $c_1 = 0$, and $c_2 = 4$. So, the solution to our initial value problem is:

$$\mathbf{x}(t) = 4e^t \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin 2t \right).$$