Math 2280 - Lecture 20

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Today we'll learn about a method for solving systems of differential equations, the *method of elimination*, that is very similar to the elimination methods we learned about in linear algebra. We'll extend this analogy further by learning about polynomial differential operators, and how we can apply analogues of Cramer's rule using these differential operators to solving systems of differential equations.

This lectures corresponds with section 4.2 of the textbook. The assigned problems for this section are:

Section 4.2 - 1, 10, 19, 28

The Method of Elimination

For systems of differential equations, particularly linear systems, we can sometimes combine equations like we do in linear algebra to eliminate dependent variables. This simplifies the equation, and eventually can be used to reduce the equation to a single linear differential equations (usually not of first-order) that we can then solve using the methods from chapter 3.

Example - Use the method of elimination to find the solution to the initial value problem:

$$x' = -3x + 2y \quad y' = -3x + 4y;$$
$$x(0) = 1, y(0) = -1$$

Solution - If we subtract the equation for x' from that for y' we get

$$y' - x' = 2y \Rightarrow x' = y' - 2y.$$

Plugging this into the given equation for x' we get:

$$y' - 2y = -3x + 2y.$$

If we differentiate both sides of this equation we get:

$$y'' - 2y' = -3x' + 2y'.$$

Again substituting $x' = y' - 2y$ we get:

$$y'' - 2y' = -3y' + 6y + 2y'.$$

We can rewrite this equation as:

$$y'' - y' - 6y = 0.$$

The characteristic equation for this ODE is $r^2 - r - 6 = (r - 3)(r + 2)$. So, its roots are $r = 3, -2$ and the general solution is:

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}.$$

If we differentiate the above equation and use our relation for y' we get:

$$y' = 3c_1e^{3t} - 2c_2e^{-2t} = -3x + 4c_1e^{3t} + 4c_2e^{-2t}$$
$$\Rightarrow -3x = -c_1e^{3t} - 6c_2e^{-2t} \Rightarrow x(t) = \frac{c_1}{3}e^{3t} + 2c_2e^{-2t}.$$

Plugging in our initial conditions we get:

$$y(0) = c_1 + c_2 = -1,$$
$$x(0) = \frac{c_1}{3} + 2c_2 = 1.$$

Solving for c_1 and c_2 we get $c_1 = -\frac{9}{5}$ and $c_2 = \frac{4}{5}$. So,

$$x(t) = -\frac{3}{5}e^{3t} + \frac{8}{5}e^{-2t}$$
$$y(t) = -\frac{9}{5}e^{3t} + \frac{4}{5}e^{-2t}.$$

Polynomial Differential Operators

A polynomial differential operator is a map from functions to functions of the form:

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0,$$

where D represents the derivative operator, and the a_i are constants. So, for example, the differential operator

$$L = D^2 + 2D - 3$$

when applied to the function

$$f(x) = x^2 + 4x - 5$$

yields

$$\begin{aligned} L(f) &= D^2(x^2 + 4x - 5) + 2D(x^2 + 4x - 5) - 3(x^2 + 4x - 5) \\ &= (2) + (4x + 8) - (3x^2 + 12x - 15) = -3x^2 - 8x + 25. \end{aligned}$$

Polynomial differential operators commute. So, if we have two differential operators, L_1 and L_2, then $L_2 L_1(f) = L_1 L_2(f)$. Note that this is not generally the case for all differential operators. It's not even necessarily the case if the a_i terms are variables instead of constants.

Any system of two linear differential equations with constant coefficients can be written in the form

$$\begin{array}{ccccc} L_1 x & + & L_2 y & = & f_1(t) \\ L_3 x & + & L_4 y & = & f_2(t) \end{array}$$

If we act on the top row with the operator L_3, and on the bottom row with the operator L_1 we get

$$\begin{array}{rcrcl} L_3L_1x & + & L_3L_2y & = & L_3f_1(t) \\ L_1L_3x & + & L_1L_4y & = & L_1f_2(t) \end{array}.$$

If we then subtract the first equation from the second, using the fact that the operators commute, we get

$$(L_1L_4 - L_2L_3)y = L_1f_2 - L_3f_1,$$

in the single variable y. Alternatively, we could have eliminated y in a like manner from the original system and obtained the equation

$$(L_1L_4 - L_2L_3)x = L_4f_1 - L_2f_2.$$

Note that the same operator, $(L_1L_4 - L_2L_3)$, appears on the left hand side of both equations. This operator is called the operational determinant:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = L_1L_4 - L_2L_3,$$

and the above two equalities are the operational versions of Cramer's rule:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1 & L_2 \\ f_2 & L_4 \end{vmatrix},$$

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1 \\ L_3 & f_2 \end{vmatrix}.$$

If the operational determinant is identically zero there can be either no solution, or infinitely many solutions, to the ODE. Also, this approach that we've applied here for systems with two variables generalizes to systems with an arbitrary numbers of variables, although the computations involved for Cramer's rule become more and more narsty.

Example - Show that the following system is degenerate, and then determine whether it has infinitely many solutions or no solution.

$$\begin{array}{rcrcl} (D+2)x & + & (D+2)y & = & e^{-3t} \\ (D+3)x & + & (D+3)y & = & e^{-2t} \end{array}$$

Solution - The operational determinant is

$$(D+2)(D+3) - (D+2)(D+3) = 0.$$

So, the system is degenerate. If we subtract the first equation from the second we get:

$$x + y = e^{-2t} - e^{-3t}.$$

Solving this for y we get:

$$y(t) = e^{-2t} - e^{-et} - x(t).$$

So, for *any* function $x(t)$ we can find a function $y(t)$ that satisfies this system. Therefore, there are infinitely many solutions.

Example - Same instructions as the previous problem. So, show the following system is degenerate, and then determine whether it has infinitely many solutions or no solution.

$$\begin{array}{rcccl} (D^2+D)x & + & D^2y & = & 2e^{-t} \\ (D^2-1)x & + & (D^2-D)y & = & 0 \end{array}$$

Solution - Here the operational determinant is:

$$(D^2+D)(D^2-D) - D^2(D^2-1) = D^4 - D^3 + D^3 - D^2 - D^4 + D^2 = 0.$$

So, again, the system is degenerate. If we subtract the second equation from the first we get:

$$(D+1)x + Dy = 2e^{-t}.$$

Differentiating both sides we get:

$$(D^2+D)x + D^2y = -2e^{-t}.$$

The first equation above is:

$$(D^2+D)x + D^2y = 2e^{-t}.$$

As $2e^{-t} \neq -2e^{-t}$ there are no solutions.