

Math 2280 - Lecture 1: Differential Equations - What Are They, and What Do They Want?

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Newton's fundamental discovery, the one which he considered necessary to keep secret and published only in the form of an anagram, consists of the following: *Data aequatione quocumque fluentes quantitates involvente fluxiones invenire et vice versa*. In contemporary mathematical language, this means: "It is useful to solve differential equations".

In other words, differential equations are a big deal! One could make a reasonable argument that they are the governing equations of the universe. Unfortunately, the universe is a complicated place, and differential equations are not, in general, easy to solve.

But take heart, just because they're hard doesn't mean they're impossible, and many fairly simple and solvable differential equations do a decent job of modeling a lot of natural phenomena. Today, we'll talk a little bit about what differential equations are, and how we solve them.

Today's lecture corresponds with section 1.1 from the textbook, and the assigned problems from this section are:

Section 1.1 - 1, 12, 15, 20, 45

The Basics

A major goal in algebra, if not the major goal, is solving algebraic equations. We want to know the values of x that solve an equation like

$$x^2 - 3x + 2 = 0.^1$$

With a differential equation, we're given a relation between a function and its derivatives, and we're asked to figure out what the function is. For example, we might be asked to solve

$$y' = y.^2$$

What we want to know is the *function* $y = f(x)$ that satisfies the above relation. Now, there may be (and very frequently are) quite a few functions that will satisfy a given differential equation, but usually they will all have the same basic form. In general we'll need more information, a.k.a. initial conditions, to specify a particular solution.

Now, this is nothing new. You remember from calculus that if I tell you

$$\frac{dy}{dx} = 2x$$

that the solution is

$$y(x) = x^2 + C.$$

There's that $+C$ term in the answer, and any value of C will solve the differential equation. If we're also told the additional information $y(0) = 3$, then this specifies the value of the constant, and there is one and only one solution to our differential equation where $y(0) = 3$, namely

$$y(x) = x^2 + 3.$$

¹The solutions are $x = 1$ and $x = 2$, by the way.

²The solution is $y = Ce^x$, but we'll get to that in a little bit.

Example - Verify that the differential equation

$$y' + y = 0$$

is solved by any equation of the form $y(x) = Ce^{-x}$, and find the value of C such that the solution satisfies $y(0) = 2$.

Solution - The derivative of $y(x) = Ce^{-x}$ is $y'(x) = -Ce^{-x}$. If we plug these into the left side above we get:

$$y' + y = -Ce^{-x} + Ce^{-x} = 0.$$

So, for any value of C , the function $y(x) = Ce^{-x}$ satisfies the above differential equation.

If we plug in $x = 0$ we get $y(0) = Ce^{-0} = C$. So, if $y(0) = 2$ we have $C = 2$, and our solution is $y(x) = 2e^{-x}$.³

We call the value of $y(0)$ the *initial condition* of our system. Technically speaking, we don't have to specify $y(0)$. We could just as well specify $y(3)$. But, for philosophical and practical reasons it's usually the value at 0 that is specified.⁴

Now, not all differential equations have solutions. The differential equation

$$(y')^2 + y^2 = -1$$

has no (real valued) solution, while the differential equation

$$(y')^2 + y^2 = 0$$

has only one solution, namely $y = 0$. Also, frequently we're able to prove a solution to a differential equation exists, but it can be extremely difficult to find it.

³This is, in fact, the *only* solution to the above differential equation with the specified initial condition, but we'll get to that a little later.

⁴Mathematically speaking, if we know how to solve it when the value at 0 is specified, we can solve it when the value anywhere else is specified. It just involves some (frequently tedious) calculating.

Terminology

The *order* of a differential equation is the order of the highest derivative that appears in it. So, the differential equation

$$y' + y = 0$$

is first-order, while

$$(y'')^2 + y^2 = 0$$

is second-order. The differential equation

$$y^{(4)} + x^2y^{(3)} + x^5y = \sin x$$

is fourth-order. For derivatives higher than the second we denote it with an exponent inside parentheses, as in the example above. So, $y^{(8)}$ would be an eighth⁵ derivative.

A *solution* to a differential equation *on the interval I* is a continuous function $u(x)$ where the derivatives $u', u'', \dots, u^{(n)}$ exist⁶ on I , and satisfy the differential equation for all x in I .

An *ordinary* differential equation is a differential equation in only one variable. We'll be almost exclusively dealing with these in this class, although we'll see some *partial* differential equations towards the end. A partial differential equation is when we have a function of more than one variable, and there is a relation between its partial derivatives.

If we're given initial values for a differential equation, a solution to the *initial value problem* will be a function that satisfies the differential equation in an interval around the initial value, and is equal to the initial value at the appropriate point.

Example - Given the solution $y(x) = \frac{1}{C-x}$ of the differential equation $y' = y^2$ solve the initial value problem

⁵yikes!

⁶where n is the order of the differential equation.

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

Solution - We need to find a value for C such that $y(1) = 2$. This just involves solving the equation:

$$\frac{1}{C-1} = 2$$

for C . This is a simple algebra problem, and its solution is $C = \frac{3}{2}$. So, the solution to the initial value problem is:

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$