Math 2280 - Lecture 17

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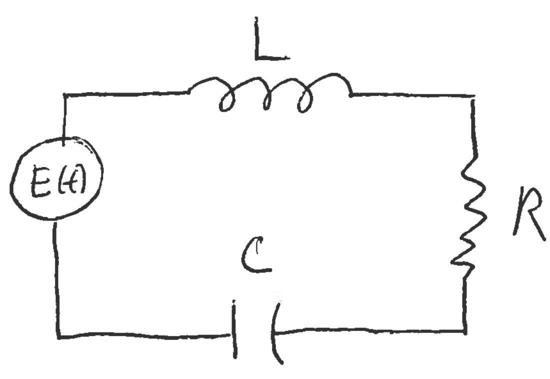
In today's lecture we'll talk about another *very* common physical system that comes up all the time in engineering - a closed circuit with a resistor, capacitor, and inductor. We'll learn that, even though physically this system is very different than a mass on a spring, the differential equation that describes them is, essentially, the same. Well, the same equations have *the same* solutions, and we'll see that the solutions we determined for the mass-spring system have their exact analogs for circuits.

Today's lecture corresponds with section 3.7 of the textbook. The assigned problems are:

Section 3.7 - 1, 5, 10, 17, 19

Electrical Circuits

For an electrical circuit of the type pictured below:



Kirchoff's second law tells us that the sum of the voltage drops across each component must equal 0:

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

This is a second order linear ODE with constant coefficients! So everybody chill out, we've got this. For example, if

$$E(t) = E_0 \sin(\omega t),$$

if we differentiate both sides of the equation we get:

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t.$$

The homogeneous solution to this will be:

$$y_h = e^{-\frac{Rt}{2L}} \left(c_1 e^{\frac{\sqrt{R^2-4L/C}}{2L}t} + c_2 e^{-\frac{\sqrt{R^2-4L/C}}{2L}t} \right).$$

This gives us a solution for I_{tr}, the *transient* current that will die out exponentially.

The particular solution will give us another term called the *steady periodic* current. It won't die out exponentially. If we run through the math, which is exactly the same as in the mechanical system, we get:

$$I_{sp}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where

$$\alpha = \arctan\left(\frac{\omega RC}{1 - LC\omega^2}\right).$$

The quantity in the denominator of our steady periodic current is denoted by the variable Z and is called the *impedence* of the circuit. The term $\omega L - 1/(\omega C)$ is called the *reactance*.

If we use the letter Z to denote the impedence, so

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

then the amplitude of our steady periodic current is

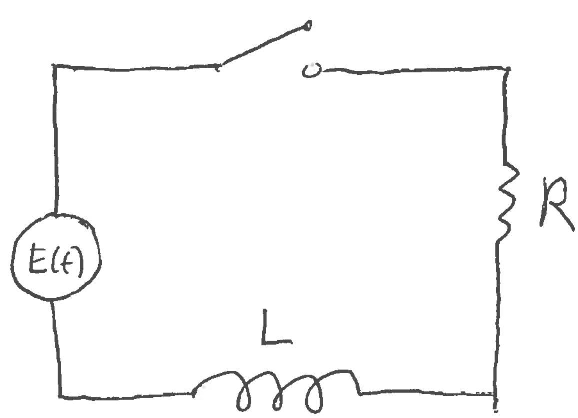
$$I_0 = \frac{E_0}{Z}.$$

If $R \neq 0$ then $Z \neq 0$, and we see that this amplitude is maximized when Z is minimized. The frequency ω that minimizes the impedence will be the frequency that makes the reactance 0. Specifically,

$$\omega_m^2 = \frac{1}{LC}.$$

This frequency is called the *resonant frequency* of the circuit.

Example - In the circuit below, suppose that $L = 2, R = 40, E(t) = 100e^{-10t}$, and $I(0) = 0$. Find the maximum current for $t \geq 0$.



Solution - If we plug the constants $L = 2, R = 40, C = 0$, and the function $E(t) = 100e^{-10t}$ into our circuit equation we get:

$$2I'(t) + 40I(t) = 100e^{-10t},$$

which simplifies to

$$I' + 20I = 50e^{-10t}.$$

This is a first-order linear differential equation, and we can solve it by multiplying both sides by the integrating factor:

$$\rho(t) = e^{\int 20dt} = e^{20t}.$$

If we do this our ODE becomes:

$$e^{20t}I'(t) + 20e^{20t}I(t) = 50e^{10t}.$$
$$\Rightarrow \frac{d}{dt}\left(Ie^{20t}\right) = 50e^{10t}.$$

Integrating both sides of the equation above gives us:

$$Ie^{20t} = \int 50e^{10t}dt = 5e^{10t} + C.$$

Solving this for the current $I(t)$ we get:

$$I(t) = 5e^{-10t} + Ce^{-20t}.$$

We have $I(0) = 0$, and we can use this to solve for C:

$$I(0) = 0 = 5 + C \Rightarrow C = -5.$$

So, the equation for our current is:

$$I(t) = 5e^{-10t} - 5e^{-20t}.$$

We want to find the value of t for which this is maximized. This will occur when the derivative is equal to 0:

$$\begin{aligned} I'(t) = -50e^{-10t} + 100e^{-20t} = 0 \\ \Rightarrow 2 = e^{10t} \\ \Rightarrow \frac{\ln 2}{10} = t. \end{aligned}$$

So, the maximum current will be:

$$I_{max}(t) = 5e^{-\ln 2} - 5e^{-2\ln 2} = \frac{5}{2} - \frac{5}{4} = \frac{5}{4} \text{ amps.}$$

Note that we can use the second derivative test to make sure this time is, indeed, a maximum:

$$\begin{aligned} I''(t) = 500e^{-10t} - 2000e^{-20t}, \\ I''\left(\frac{\ln 2}{10}\right) = 500e^{-\ln 2} - 2000e^{-2\ln 2} = 250 - 500 = -250 < 0. \end{aligned}$$

Example - The parameters of an RLC circuit with input voltage $E(t)$ are:

$$R = 30\Omega,\ L = 10H,\ C = 0.02F;\ E(t) = 50\sin(2t)V.$$

Substitute

$$I_{sp}(t) = A\cos\omega t + B\sin\omega t$$

using the appropriate value of ω to find the steady periodic current in the form $I_{sp}(t) = I_0\sin(\omega t - \delta)$.

Solution - The angular frequency of our driving function is 2, so $\omega = 2$, and the amplitude I_0 will be:

$$I_0 = \frac{E_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{50}{\sqrt{900 + (20 - 25)^2}} = \frac{50}{\sqrt{925}} = \frac{10}{\sqrt{37}}.$$

As for α:

$$\alpha = arctan\left(\frac{2(30)(.02)}{1 - (10)(.02)(2^2)}\right) = arctan(6).$$

So,

$$\begin{aligned} I_{sp} &= \frac{10}{\sqrt{37}}\cos(2t - arctan(6)) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - arctan(6) + \frac{\pi}{2}\right) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - (arctan(6) - \frac{\pi}{2})\right) \\ &= \frac{10}{\sqrt{37}}\sin\left(2t - (\frac{3\pi}{2} + arctan(6))\right). \end{aligned}$$