

# Math 2280 - Lecture 17

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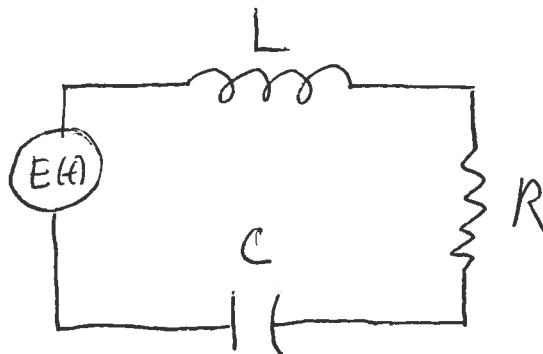
In today's lecture we'll talk about another *very* common physical system that comes up all the time in engineering - a closed circuit with a resistor, capacitor, and inductor. We'll learn that, even though physically this system is very different than a mass on a spring, the differential equation that describes them is, essentially, the same. Well, the same equations have *the same* solutions, and we'll see that the solutions we determined for the mass-spring system have their exact analogs for circuits.

Today's lecture corresponds with section 3.7 of the textbook. The assigned problems are:

Section 3.7 - 1, 5, 10, 17, 19

## Electrical Circuits

For an electrical circuit of the type pictured below:



Kirchoff's second law tells us that the sum of the voltage drops across each component must equal 0:

$$L \frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

This is a second order linear ODE with constant coefficients! So everybody chill out, we've got this. For example, if

$$E(t) = E_0 \sin(\omega t),$$

if we differentiate both sides of the equation we get:

$$LI'' + RI' + \frac{1}{C}I = \omega E_0 \cos \omega t.$$

The homogeneous solution to this will be:

$$y_h = e^{-\frac{Rt}{2L}} \left( c_1 e^{\frac{\sqrt{R^2 - 4L/C}t}{2L}} + c_2 e^{-\frac{\sqrt{R^2 - 4L/C}t}{2L}} \right).$$

This gives us a solution for  $I_{tr}$ , the *transient* current that will die out exponentially.

The particular solution will give us another term called the *steady periodic* current. It won't die out exponentially. If we run through the math, which is exactly the same as in the mechanical system, we get:

$$I_{sp}(t) = \frac{E_0 \cos(\omega t - \alpha)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

where

$$\alpha = \arctan \left( \frac{\omega RC}{1 - LC\omega^2} \right).$$

The quantity in the denominator of our steady periodic current is denoted by the variable  $Z$  and is called the *impedance* of the circuit. The term  $\omega L - 1/(\omega C)$  is called the *reactance*.

If we use the letter  $Z$  to denote the impedance, so

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

then the amplitude of our steady periodic current is

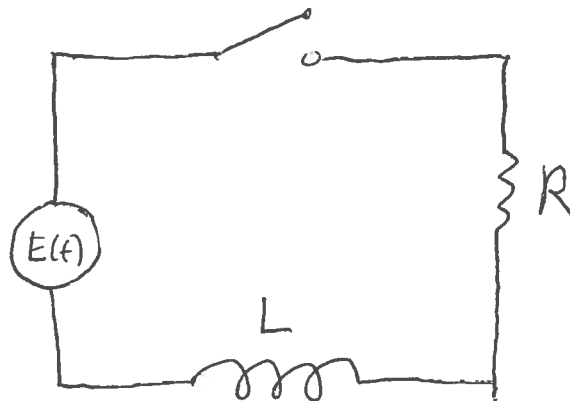
$$I_0 = \frac{E_0}{Z}.$$

If  $R \neq 0$  then  $Z \neq 0$ , and we see that this amplitude is maximized when  $Z$  is minimized. The frequency  $\omega$  that minimizes the impedance will be the frequency that makes the reactance 0. Specifically,

$$\omega_m^2 = \frac{1}{LC}.$$

This frequency is called the *resonant frequency* of the circuit.

*Example* - In the circuit below, suppose that  $L = 2$ ,  $R = 40$ ,  $E(t) = 100e^{-10t}$ , and  $I(0) = 0$ . Find the maximum current for  $t \geq 0$ .



More room for the example problem.

*Example* - The parameters of an RLC circuit with input voltage  $E(t)$  are:

$$R = 30\Omega, L = 10H, C = 0.02F; E(t) = 50 \sin(2t)V.$$

Substitute

$$I_{sp}(t) = A \cos \omega t + B \sin \omega t$$

using the appropriate value of  $\omega$  to find the steady periodic current in the form  $I_{sp}(t) = I_0 \sin(\omega t - \delta)$ .