Math 2280 - Lecture 16

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In today's lecture we'll return to our mass-spring mechanical system example, and examine what happens when there is a periodic driving force $f(t) = F_0 \cos \omega t$.

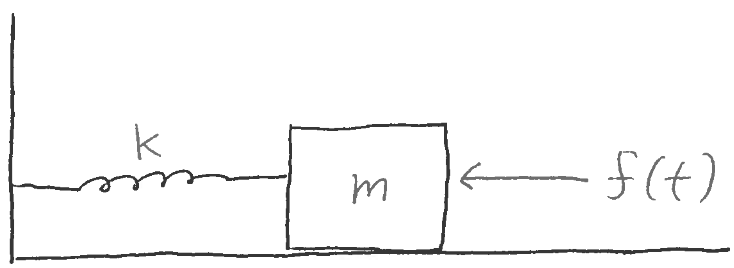
This lecture corresponds with section 3.6 of the textbook, and the assigned problems are:

Section 3.6 - 1, 2, 9, 17, 24

Forced Oscillations

In this lecture we'll delve deeper into the simple mechanical system we examined two lectures ago, and discuss some of the consequences of adding a forcing function to the system.

Suppose we have a spring-mass system with an external driving force, pictured schematically below:



Assuming there is no damping, we can model this system by a differential equation of the form:

$$mx'' + kx = f(t)$$

Now, suppose our forcing function is of the form $f(t) = F_0 \cos \omega t$, where $\omega \neq \sqrt{k/m}$. Then, the method of undetermined coefficients would lead us to guess a particular solution of the form:

$$x(t) = A \cos \omega t + B \sin \omega t.$$

If we plug this guess into our differential equation we get the relation:

$$-Am\omega^2 \cos \omega t + Ak \cos \omega t - Bm\omega^2 \sin \omega t + Bk \sin \omega t = F_0 \cos \omega t,$$

which if we solve for the constants A and B we get:

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2},$$
$$B = 0.$$

Consequently, our particular solution will be:

$$x_p(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t.$$

And, in general, our solution will be of the form:

$$x(t) = \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t + c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.$$

We can, equivalently, rewrite the above solution as

$$x(t) = C \cos (\omega_0 t - \alpha) + \left(\frac{F_0/m}{\omega_0^2 - \omega^2} \right) \cos \omega t,$$

just as we did for the undamped case examined two lectures ago.

Example - Express the solution to the initial value problem

$$x'' + 9x = 10\cos 2t,$$
$$x(0) = x'(0) = 0,$$

as a sum of two oscillations as in the equation above.

Solution - The particular solution x_p will be a combination of sine and cosine terms of the form:

$$x_p = A\cos 2t + B\sin 2t.$$

Taking derivatives and plugging these into the ODE we get:

$$x_p' = -2A\sin 2t + 2B\cos 2t,$$
$$x_p'' = -4A\cos 2t - 4B\sin 2t,$$

and so,

$$x_p'' + 9x_p = 5A\cos 2t + 5B\sin 2t = 10\cos 2t.$$

So, $A = 2, B = 0$, and our particular solution is $x_p = 2\cos 2t$.

The homogeneous equation has characteristic equation $r^2 + 9$, and so the homogeneous solution is of the form:

$$x_h = c_1 \sin 3t + c_2 \cos 3t.$$

Our solution will be $x = x_h + x_p$, and if we plug in our initial conditions we get:

$$x(0) = c_2 + 2 = 0,$$
$$x'(0) = 3c_1 = 0,$$

from which we get $c_1 = 0$ and $c_2 = -2$. Therefore, our solution is:

$$x(t) = 2\cos 2t - 2\cos 3t.$$

Beats

If we impose the initial conditions: $x(0) = x'(0) = 0$ then we have:

$$c_1 = 0$$

and

$$c_2 = -\frac{F_0/m}{\omega_0^2 - \omega^2}.$$

Plugging these in to our solution we get:

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t).$$

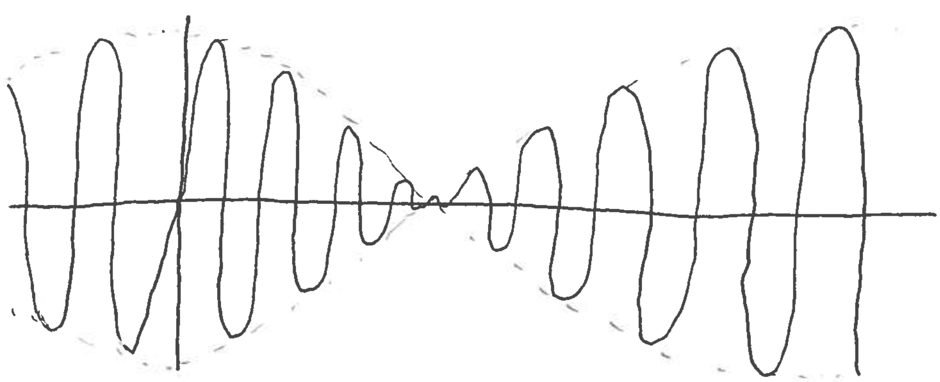
If we use the relation

$$2 \sin A \cos B = \cos (A - B) - \cos (A + B)$$

we can rewrite the above equation as:

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)}{2}t\right) \cos\left(\frac{(\omega_0 + \omega)}{2}t\right).$$

Now, if $\omega_0 \approx \omega$, this solution looks like a higher frequency wave oscillating within a lower frequency envelope:



This is a situation known as beats.

Resonance

What if $\omega = \omega_0$? Then, for our particular solution we'd guess:

$$x_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t).$$

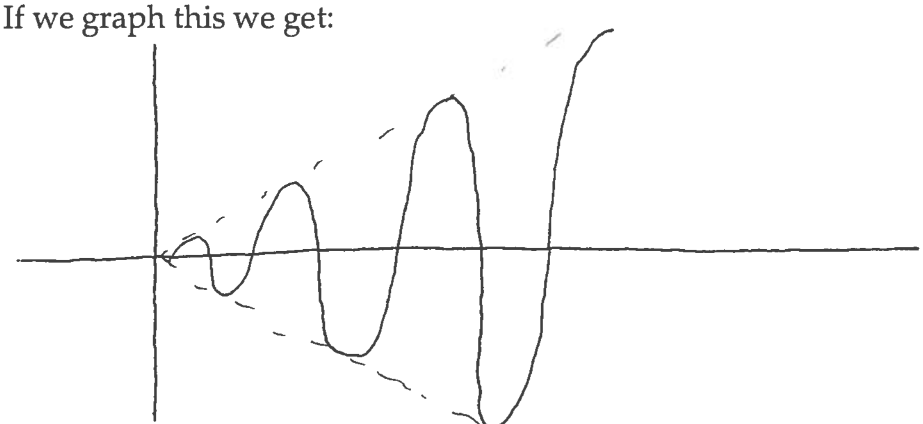
If we make this guess and work it out with the initial conditions $x(0) = x'(0) = 0$ we get:

$$A = 0$$
$$B = \frac{F_0}{2m\omega_0}$$

with corresponding particular solution:

$$x_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t).$$

If we graph this we get:



This is a situation known as resonance.