Math 2280 - Lecture 15

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In the last two lectures we're learned how to solve homogeneous linear differential equations with constant coefficients. In other words, equations of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_i \in \mathbb{R}$. We've also looked at a mechanical application of these types of equations; the mass-spring-dashpot system.

Today, we're going to talk about nonhomogeneous linear differential equations with constant coefficients. So, equations of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $f(x) \neq 0$. We'll learn how to solve these in particular cases that come up frequently, and we'll touch upon the idea behind the general solution.

Today's lecture corresponds with section 3.5 of the textbook, and the exercises for this section are

Section 3.5 - 1, 11, 23, 28, 35, 47, 56.

Note that I've assigned more problems than usual for this section because I think the best way to understand this material is to work a number of problems, as compared to memorizing theorems and concepts.

Nonhomogeneous Linear Differential Equations with Constant Coefficients

Up to this point we've dealt almost exclusively with homogeneous linear equations:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Now, we're going to take a look at what we get when we add a "driving force", a.k.a. a nonhomogeneous term, to the right side of our equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

We've learned that, in general, if we have a homogeneous nth-order linear differential equation (whether with constant coefficients or not) that "all" we have to do is solve the homogeneous equation to get n linearly independent homogeneous solutions:

$$y_h = c_1 y_1 + \cdots + c_n y_n,$$

and then find a particular solution to the nonhomogeneous solution y_p.[1] If you've got these, then the general solution is of the form:

$$y = y_p + y_h.$$

where the c_k in y_h are determined by initial conditions.

Finding the homogeneous solution is, in general, a very difficult thing to do, and finding a particular solutions is also, in general, a hard thing to do. However, we have learned how to find the homogeneous solution in the special case of constant coefficients. Today we'll learn a method for finding a particular solution to the nonhomogeneous equation, once we know the general solution to the corresponding homogeneous equation.

[1] Note that in the textbook sometimes the homogeneous solution is called the *complementary solution*, and is written as y_c.

The method, while it can be codified and made rigorous, is best understood as a few rules that we need to apply, and I think these rules are best understood in the context of a few examples.

First, suppose we have a differential equation of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

where $f(x)$ is a polynomial of degree m. In this situation we "guess" that our particular solution will also be a polynomial of degree m:

$$y_p = A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0,$$

and then we plug this guess in and figure out what the coefficients needs to be in order to make the guess correct.

Example - Find a particular solution to the differential equation:

$$y'' + 3y' + 4y = 3x + 2$$

Solution - We "guess" that our particular solution will be of the form: $y_p = Ax + B$. If we then plug this guess into the ODE we get:

$$\begin{aligned} y_p &= Ax + B, \\ y_p' &= A, \\ y_p'' &= 0, \end{aligned}$$

and so,

$$0 + 3A + 4(Ax + B) = 3x + 2.$$

Solving this for A and B we get $A = \frac{3}{4}$ and $B = -\frac{1}{16}$, and so our particular solution is:

$$y_p = \frac{3}{4}x - \frac{1}{16}.$$

The same basic approach works if $f(x)$ is of the form:

$$f(x) = Ce^{rx},$$

or

$$f(x) = c_1 \sin kx + c_2 \cos kx.$$

In these cases, we guess:

$$y_p = Ae^{rx},$$

and

$$y_p = A_1 \sin kx + B_1 \cos kx$$

respectively. If $f(x)$ has any of these three types of terms multiplied together, we just guess that our solution has these terms multiplied together. If $f(x)$ has any of these three types of terms added together, we just guess that our solution has these terms added together. Make sure to include all the polynomial terms!

So, for example, if we want to guess the form of the particular solution to the ODE:

$$y^{(3)} + 9y' = x \sin x + x^2 e^{2x}$$

we would guess:

$$y_p = A \cos x + B \sin x + Cx \cos x + Dx \sin x + Ee^{2x} + Fxe^{2x} + Gx^2 e^{2x}$$

and then calculate A, B, C, D, E, F and G appropriately. This is called the method of undetermined coefficients.

Clear as mud? I promise it's not that bad once you get the hang of it. Let's work a few examples.

Example - Find a particular solution to the ODE

$$y'' - y' - 6y = 2\sin 3x.$$

Solution - As our inhomogeneous terms is a sum of sines and cosines, our "guess" will be as well:

$$y_p = A\sin 3x + B\cos 3x.$$

Plugging this into our ODE we get:

$$-9A\sin 3x - 9B\cos 3x - 3A\cos 3x + 3B\sin 3x - 6A\sin 3x - 6B\cos 3x = 2\sin 3x.$$

Grouping the sine and cosine terms together we get:

$$(-15A + 3B)\sin 3x + (-15B - 3A)\cos 3x = 2\sin 3x.$$

This gives us the linear equations:

$$-15A + 3B = 2$$
$$-3A - 15B = 0.$$

Solving these for A and B we get $A = -\frac{5}{39}$, and $B = \frac{1}{39}$. So, our particular solution is:

$$y_p = -\frac{5}{39}\sin 3x + \frac{1}{39}\cos 3x.$$

Example - Find a particular solution of the ODE:

$$y'' - 4y = 2e^{3x}.$$

Solution - Our inhomogeneous term is $2e^{3x}$, so we'll "guess" our particular solution is of the form:

$$y_p = Ae^{3x}.$$

Plugging this into the ODE we get:

$$9Ae^{3x} - 4Ae^{3x} = 2e^{3x}.$$

So, $5A = 2$, or $A = \frac{2}{5}$, and we get the particular solution:

$$y_p = \frac{2}{5}e^{3x}.$$

Now, there's a potential snag that we occasionally hit with this method. We can illustrate this with an example. Suppose we want to solve the ODE

$$y'' - 4y = 2e^{2x},$$

which is superficially quite similar to our last example. If we try $y_p = Ae^{2x}$ we get:

$$4Ae^{2x} - 4Ae^{2x} = 2e^{2x}$$

which won't work, as this is just $0 = 2e^{2x}$. Uh oh! What do we do? The problem here is that our guess is not linearly independent of our homogeneous solutions. What we do in this case is we just multiply our guess by x until we get a linearly independent guess.

In this case we would guess $y_p = Axe^{2x}$ and then solve for A to get $y_p = (1/2)xe^{2x}$, making the general form of our solution:

$$y = \frac{1}{2}xe^{2x} + c_1e^{2x} + c_2e^{-2x}.$$

So, if any of the terms in our "guess" are not linearly independent of the solutions to the homogeneous equation we multiply the corresponding terms in the guess by x until they are. Let's do another example.

Example - Find a particular solution to

$$y^{(3)} + y'' = 3e^x + 4x^2.$$

Solution - As the inhomogeneous term is the sum of an exponential and a second-degree polynomial, we'd first "guess" that our particular solution is of the form:

$$y_p = Ae^x + Bx^2 + Cx + D.$$

However, the characteristic equation for the corresponding homogeneous ODE is $r^2(r+1)$, which has solutions $e^{-x}, 1$, and x. So, the term $Cx + D$ in our "guess" won't work, and we need to multiply the polynomial part by x^2 to make all the terms linearly independent:

$$y_p = Ae^x + Bx^4 + Cx^3 + Dx^2.$$

If we plug this into our ODE we get:

$$Ae^x + 24Bx + 6C + Ae^x + 12Bx^2 + 6Cx + 2D = 3e^x + 4x^2.$$

From this we get $2A = 3, 12B = 4, 24B + 6C = 0$, and $6C + 2D = 0$. Solving these we get $A = \frac{3}{2}, B = \frac{1}{3}, C = -\frac{4}{3}$, and $D = 4$. So, our particular solution is:

$$y_p = \frac{3}{2}e^x + \frac{1}{3}x^4 - \frac{4}{3}x^3 + 4x^2.$$

Finally, we need to mention that there's a method that works, in general, to find a particular solution to *any* linear ODE as long as the general solution to the homogeneous equation is known. The book touches upon this for the second-order case. We won't get into the details here, but I feel I should at least go over the formula and do an example problem.

Theorem : *Variation of Parameters* - If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ as the general solution to the associated homogeneous equation, then a particular solution is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx,$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Example - Use the method of variation of parameters to find a particular solution to the given differential equation

$$y'' - 4y' + 4y = 2e^{2x}.$$

Soltuion - The corresponding homogeneous equation has the characteristic equation $r^2 - 4r + 4 = (r-2)^2$. So,it has one root, $r = 2$, of multiplicity 2, and the two linearly independent solutions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$. If we calculate the Wronskian of these two functions we get:

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}.$$

So, the corresponding integrals are:

$$\int \frac{y_2 f}{W}dx = \int \frac{xe^{2x}(2e^{2x})}{e^{4x}}dx = \int 2x dx = x^2,$$

$$\int \frac{y_1 f}{W}dx = \int \frac{e^{2x}(2e^{2x})}{e^{4x}}dx = \int 2dx = 2x,$$

and the particular solution is:

$$y_p = -y_1 \int \frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx = -e^{2x} x^2 + x e^{2x}(2x) = x^2 e^{2x}.$$