Math 2280 - Lecture 14

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Summer 2013

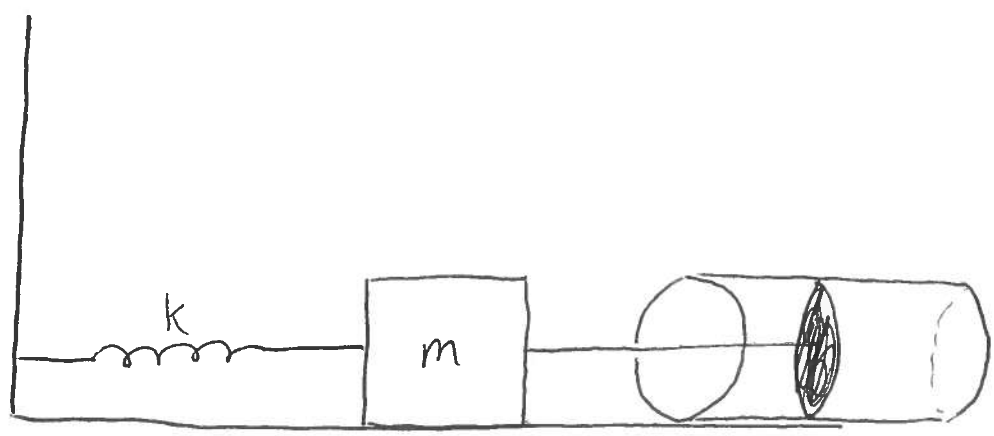
In today's lecture we're going to examine, in detail, a physical system whose behavior is modeled by a second-order linear ODE with constant coefficients. We'll examine the different possible solutions, what determines these solutions, and what these solutions mean as far as the behavior of the system is concerned.

The assigned problems for this section are:

Section 3.4 - 1, 5, 18, 21

Simple Mechanical Systems and the Differential Equations that Love Them

Today we're going to examine a fairly simple mechanical system in detail, and look closely at its possible solutions.



We have a mass on a spring connected to a dashpot. The forces on the mass are:

The force from the spring:

$$F_S = -kx.$$

The force from the dashpot:

$$F_R = -cv.$$

An external driving force:

$$F_E = f(t).$$

Today we'll assume that $f(t) = 0$. The inhomogeneous, $f(t) \neq 0$, situation we'll examine in detail next week.

According to Newton's second law:

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx.$$

Or, after a little algebra,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

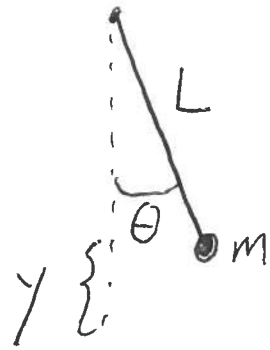
This is a second-order linear homogeneous ODE with constant coefficients. We can rewrite this as:[1]

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Before solving this, let's take a look at another basic mechanical example; the simple pendulum.

[1] Just diving everything by the mass m.

Bad Drawing:



We can apply the conservation of energy to the pendulum to derive the differential equation:

$$mgy + \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 = C.$$

If we note that $y = L(1 - \cos\theta)$ we get:

$$mgL(1 - \cos\theta) + \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 = C.$$

Differentiating both sides of this we get the equation:

$$mgL\sin\theta\frac{d\theta}{dt} + mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) = 0.$$

Dividing through by the common factors we get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

This is not a linear ODE. However, if we assume θ is small we can use the approximation $\sin\theta \approx \theta$ to get:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

This is, essentially, the same equation we saw before with the mass-spring-dashpot system if we set $c = 0$. A fundamental idea in physics is that the same equations have the same solutions, and so the behavior we witness for the mass-spring-dashpot system will be analogous to the behavior of the pendulum.

The Solutions and What They Mean

The differential equation for the pendulum above has the solutions:

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right).$$

If we choose:

$$A = \sqrt{c_1^2 + c_2^2}$$

and

$$\cos\phi = \frac{c_1}{A}, \sin\phi = \frac{c_2}{A}$$

then

$$\theta(t) = A\left(\cos\phi\cos\left(\sqrt{\frac{g}{L}}t\right) + \sin\theta\sin\left(\sqrt{\frac{g}{L}}t\right)\right).$$

If we use the relation:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

we can rewrite $\theta(t)$ as:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t - \phi\right).$$

This is the simplified equation for simple harmonic motion. We call the terms:

$$A = \text{Amplitude},$$
$$\phi = \text{Phase shift},$$
$$\sqrt{\frac{g}{L}} = \text{Angular frequency} = \omega.$$

From these we define the terms:

$$\text{Frequency}: f = \frac{\omega}{2\pi},$$
$$\text{Period}: T = \frac{1}{f} = \frac{2\pi}{\omega}.$$

Example - Most grandfather clocks have pendulums with adjustable lengths. One such clock loses 10 min per day when the length of its pendulum is 30 in. With what length pendulum will this clock keep perfect time?

Now, if we look again at the mass-spring-dashpot system we examined at the beginning of this lecture we note that we can rewrite the differential equation as:

$$x'' + 2px' + \omega_0^2 x = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{m}} > 0, \text{ and } p = \frac{c}{2m} > 0.$$

If we use the quadratic equation to solve the characteristic equation for this ODE we get:

$$\frac{-2p \pm \sqrt{(2p)^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

From this we get three fundamental possibilities, depending on the sign of the *discriminant* $p^2 - \omega_0^2$:

Case 1: Overdamped -

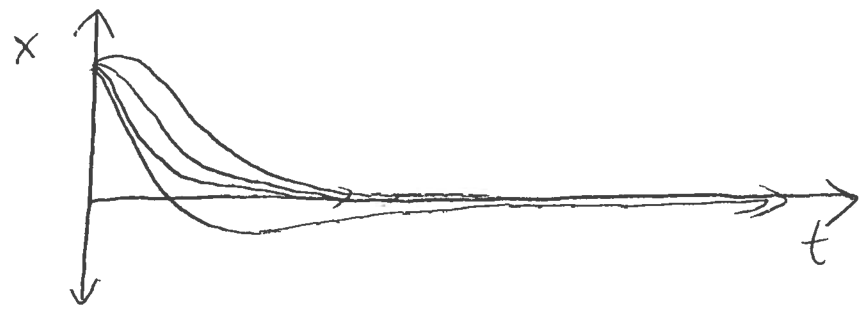
This case occurs when

$$p > \omega_0 \text{ a.k.a. } c^2 > 4mk \text{ a.k.a. the discriminant is positive.}$$

In this situation we have 2 real negative roots, and our solution is of the form:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Some representative graphs of this situation are below. We note that the solution asymptotically goes to 0 as $t \to \infty$.



Case 2: Critically Damped -

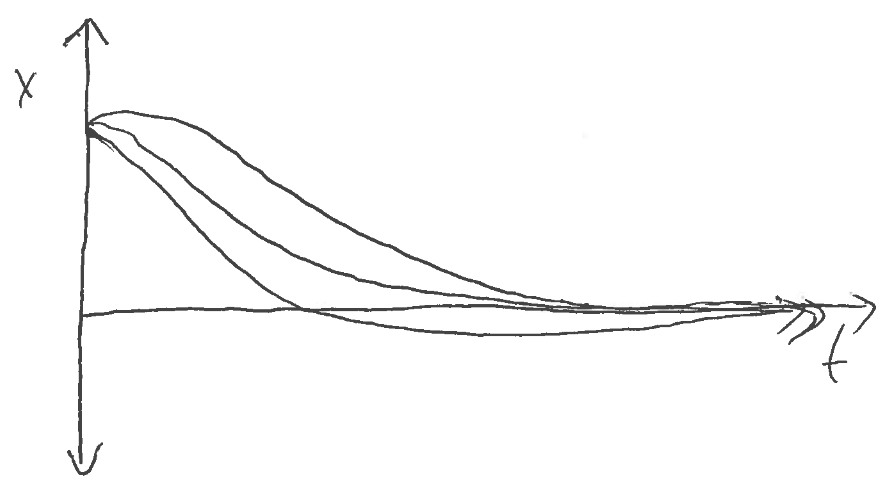
This case occurs when

$$p = \omega_0 \text{ a.k.a. } c^2 = 4mk \text{ a.k.a the discriminant is zero.}$$

In this situation we have one real negative root and our solution is of the form:

$$x(t) = e^{-rt}(c_1 + c_2 t).$$

Some representative graphs of this situation are below. We note that, again, the solution asymptotically goes to 0 as $t \to \infty$.



Case 3: Underdamped -

This case occurs when

$$p < \omega_0 \text{ a.k.a. } c^2 < 4km \text{ a.k.a. the discriminant is negative.}$$

In this situation we have two complex roots and our solution is of the form:

$$x(t) = e^{-pt}(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t))$$

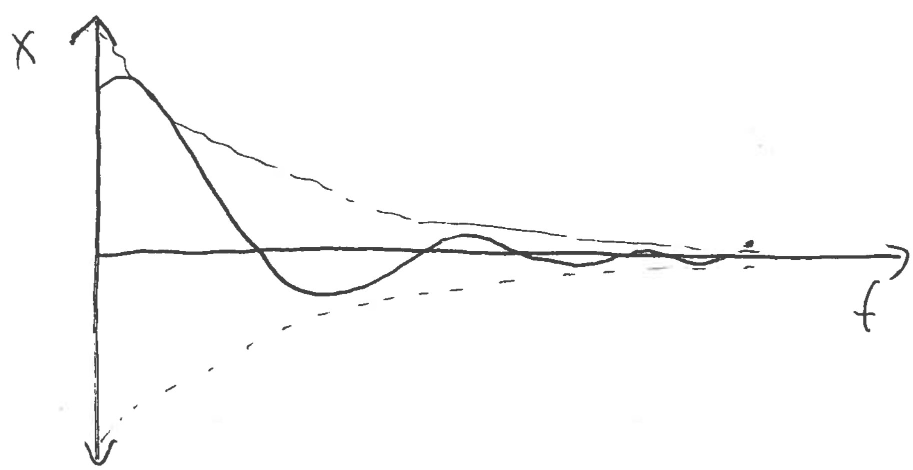
where

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km - c^2}}{2m}.$$

As explained for the pendulum we can rewrite this solution as:

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha).$$

A representative graph of this situation is below. We note, again, that the solution assymptotically approaches 0 as $t \to \infty$.[2]



[2]Unless $p = 0$, in which case we have the behavior for the pendulum we examined earlier.

Example - Solve the ODE that models the mass-spring-dashpot system with the parameters:

$$m = \frac{1}{2},\ c = 3,\ k = 4,$$

$$x_0 = 2,\ v_0 = 0.$$

Is the system overdamped, critically damped, or underdamped?