Math 2280 - Lecture 13

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Today we're going to talk about a class of nth-order linear homogeneous ODEs that we know how to solve in general. These are linear homogeneous ODEs where the coefficient functions are in fact *constant* functions. In this situation we do have a general procedure that we can use to completely solve the ODEs. Today, we'll learn what this general procedure is.

Today's lecture corresponds with section 3.3 of the textbook, and the assigned problems from this section are:

Section 3.3 - 1, 10, 25, 30, 43

The Mouthful : Linear Homogeneous Ordinary Differential Equations with Constant Coefficients

A linear homogeneous nth-order ODE with constant coefficients is a differential equation of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where the $a_i \in \mathbb{R}$ are constants, and $a_n \neq 0$.

We found out that for second-order ODEs of this type we have solutions of the form $y = e^{rx}$. Let's see if that works for higher-order ODEs. If we plug $y = e^{rx}$ into the ODE above we get:

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots a_1 r e^{rx} + a_0 e^{rx} = 0$$

If we divide both sides by e^{rx}, which we know we can do for any value of x as e^{rx} is never 0, we get the equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

which is true if and only if r is a root of the *characteristic polynomial* for our differential equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

So, to repeat, if r is a root of the characteristic polynomial for our differential equation, then e^{rx} is a solution to our differential equation.

Now, the fundamental theorem of algebra[1] says that for any polynomial:

$$x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

(we're assuming $a_n = 1$) we can factor it as:

$$(x - c_1)^{r_1} (x - c_2)^{r_2} \cdots (x - c_m)^{r_m}$$

where $r_1 + r_2 + \cdots + r_m = n$, and the constants c_i, the roots of the polynomial, could be complex numbers. This presents us with a number of possible situations.

[1]With a name like that, you know it's important.

A Number of Possible Situations

We can break down our possible situations into four distinct cases of varying complexity.

Case 1 - All roots of the characteristic polynomial are real and distinct.

If this is the case then we have n distinct solutions of the form

$$\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\},$$

and our general solution will be:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

We can check that the solutions are linearly independent by noting that the Wronskian for these solutions, evaluated at 0, is:

$$W(0) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}.$$

This is something called the "Vandemonde determinant", and it's never 0 when the r_i are distinct. This is proven in one of the suggested homework problems.

Example - Find the general solution to the differential equation

$$y^{(3)} - 6y'' - y' + 30y = 0.$$

Solution - The characteristic for this differential equation is:

$$r^3 - 6r^2 - r + 30 = 0.$$

This factors as

$$(r+2)(r-5)(r-3).$$

So, the roots are $-2, 5,$ and 3. The general solution is:

$$y(x) = c_1 e^{-2x} + c_2 e^{3x} + c_3 e^{5x}.$$

Case 2 - Repeated Roots, all real.

Let's first just look at the case when our polynomial factors as:

$$(x-c)^n.$$

In this case our differential equation can be written as:

$$(D-c)^n y = 0$$

where D is the "differential operator", an operator that when applied to a function takes the derivative of that function. It's a linear operator, and if you raise it to any power, that just means you take that number of derivatives. For example:

$$(D-1)^2 y = (D^2 - 2D + 1)y = y'' - 2y' + y.$$

Now, *any* solution y to our differential equation can be written in the form $y(x) = u(x)e^{cx}$. If we apply the operator $(D-c)$ to this solution we get:

$$(D-c)u(x)e^{cx} = u'(x)e^{cx} + cu(x)e^{cx} - cu(x)e^{cx} = u'(x)e^{cx}.$$

If we repeat this process n times we get:

$$(D-c)^n u(x)e^{cx} = u^{(n)}(x)e^{cx}.$$

If $u(x)e^{cx}$ is a solution to our differential equation then we have $u^{(n)}(x)e^{cx} = 0$, which implies $u^{(n)} = 0$. This means:

$$u(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}.$$

From this we get a set of linearly independent solutions to our ODE $\{e^{cx}, xe^{cx}, x^2e^{cx}, \ldots, x^{n-1}e^{cx}\}$.[2]

Example - Find the general solution to the differential equation

$$y^{(3)} - 4y'' - 3y' + 18y = 0.$$

Solution - The characteristic equation for this ODE is:

$$r^3 - 4r^2 - 3r + 18 = 0.$$

This polynomial factors as:

$$(r-3)^2(r+2).$$

So, the roots are $3, 3, -2$. I've repeated 3 in this list because it's a root of multiplicity 2. The general solution will be:

$$y(x) = c_2e^{-2x} + c_2e^{3x} + c_3xe^{3x}.$$

[2] It's a quick exercise to check that the Wronskian for these functions is always non-zero.

Case 3 - Complex Roots.

Suppose that the characteristic polynomial for our linear homogeneous ODE is second-order and can be factored as:

$$(x-r)(x-\overline{r})$$

where r is complex and $\overline{r}$ is its complex conjugate. Note that for any polynomial:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$$

if the a_i are real and if r is a complex root, then $\overline{r}$ must also be a complex root. So, for any characteristic equation if we get a complex root, we'll get another root by taking its complex conjugate.

OK, so if r is a complex root, then we can write r as $r = a + bi$, and we get a solution to our differential equation:

$$e^{rx} = e^{(a+ib)x} = e^{ax}e^{ibx} = e^{ax}(\cos(bx) + i\sin(bx))$$

where we've used Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.[3]$$

Now, the fact that $\overline{r} = a - ib$ is also a solution to our characteristic polynomial gives us another solution:

$$e^{\overline{r}} = e^{ax}(\cos(bx) - i\sin(bx)).$$

A linear combination of these two can be written as:

[3]Richard Feynman called this the most amazing formula in all of mathematics.

$$k_1e^{rx} + k_2e^{\overline{r}x} = e^{ax}((k_1 + k_2)\cos{(bx)} + (k_1 - k_2)i\sin{(bx)}).$$

If we allow k_1, k_2 to be complex, then we can choose them so that the above equation is equal to:

$$e^{ax}(c_1\cos{(bx)} + c_2\sin{(bx)})$$

where the coefficients c_1 and c_2 are arbitrary *real* constants. So, we get two linearly independent solutions to our ODE: $e^{ax}\cos{(bx)}$ and $e^{ax}\sin{(bx)}$, and any other solution can be written as a linear combination of these two.

Example - What is the general solution to the ODE:

$$y'' + 4y' + 8y = 0.$$

Solution - The characteristic equation for this ODE is:

$$r^2 + 4r + 8 = 0.$$

Using the quadratic equation we get that the two roots are:

$$\frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i.$$

The corresponding general solution will be:

$$y(x) = c_1e^{-2x}\sin 2x + c_2e^{-2x}\cos 2x.$$

Case 4 - The General Case

Suppose, yet again, that we have a linear homogeneous ordinary differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 = 0$$

To solve this problem, *in general*, what we do is we first calculate the roots of the characteristic polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = (x - r_1)^{k_1} (x - r_2)^{k_2} \cdots (x - r_m)^{k_m}$$

Now, there will be some set of these roots, say r_1 through r_l, which are real. For each of these we will get a number of solutions equal to the degree of the root. So, for example, for our first root r_1 we will get solutions:

$$c_1 e^{r_1 x}, c_2 x e^{r_1 x}, \ldots, c_{k_1} x^{k_1 - 1} e^{r_1 x}.$$

As for the other roots, r_{l+1} through r_m, these will come in complex conjugate pairs, and each element of these pairs will have the same degree. So, for example, for the root r_{l+1} if we arrange out terms appropriately r_{l+2} will be its complex conjugate, and we'll have $k_{l+1} = k_{l+2}$. For this pair of roots we'll get the set of solutions:

$$c_1 e^{a_{l+1} x} \cos(b_{l+1} x), c_2 x e^{a_{l+1} x} \cos(b_{l+1} x), \ldots, c_{k_{l+1}} x^{k_{l+1} - 1} e^{a_{l+1} x} \cos(b_{l+1} x)$$

and

$$d_1 e^{a_{l+1} x} \sin(b_{l+1} x), d_2 x e^{a_{l+1} x} \sin(b_{l+1} x), \ldots, d_{k_{l+1}} x^{k_{l+1} - 1} e^{a_{l+1} x} \sin(b_{l+1} x).$$

where the c_i and d_i are unknown constants, and the constants a_{l+1} and b_{l+1} are the real part and complex part, respectively, of the root $r_{l+1} = a_{l+1} + i b_{l+1}$.

Clear as mud? Yeah, in its full generality it's kind of confusing to state, but know that this more general statement just builds up from the particular cases we've seen so far, and in this class and in most situations we'll only be dealing with relatively small order ODEs, so the full machinery of the more general statement won't be necessary in practice. But, it's good to know, or at least to see.

Finally, it's worth nothing that, in practice, the hardest part of this problem for high degree ODEs is finding the roots of the characteristic polynomial. Turns out this is a difficult and *very* interesting problem, that you'll learn much more about if you take a modern algebra class.