Math 2280 - Final Exam

University of Utah

Summer 2013

Name: Solutions by Dylan Zwick

This is a two-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction. There are 200 possible points on this exam.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right).$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt.$$

1. **Basic Definitions** (15 points)

   Circle or state the correct answer to the questions about the following differential equation:

$$(x^3 + 2xe^x - \sin(x))y^{(5)} + x^2y' - e^{-3x}y = \sinh(x^3 + 2)$$

   (3 point) The differential equation is: **Linear** Nonlinear

   (3 points) The order of the differential equation is: 5

   (3 points) The corresponding homogeneous equation is:

$$(x^3 + 2xe^x - \sin(x))y^{(5)} + x^2y' - e^{-3x}y = 0$$

   For the differential equation:

$$(y')^2 = 2y + 1$$

   (3 point) The differential equation is: Linear **Nonlinear**

   (3 point) The order of the differential equation is: 1

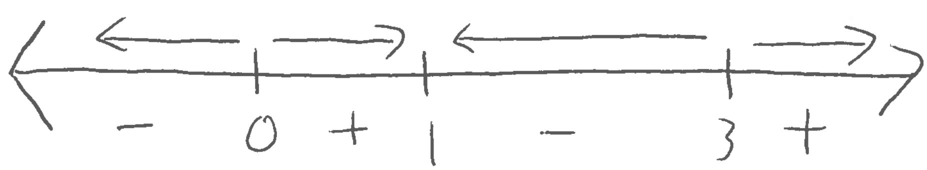
2. **Phase Diagrams** (20 points)

For the autonomous differential equation:

$$\frac{dx}{dt} = x^3 - 4x^2 + 3x$$

Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

Solution - The polynomial $x^3 - 4x^2 + 3x$ factors as $(x-3)(x-1)x$, and has roots at $x = 0, 1, 3$. The corresponding phase diagram is:



The critical points $x = 0, 3$ are unstable, while the critical point $x = 1$ is stable.

3. **First-Order Linear ODEs** (20 points)

Find the solution to the initial value problem:

$$y' + 2xy = x$$
$$y(0) = -2.$$

Solution - The integrating factor is:

$$\rho(x) = e^{\int 2x dx} = e^{x^2}.$$

Multiplying both sides by this factor we get:

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$
$$\Rightarrow \frac{d}{dx}\left(e^{x^2} y\right) = xe^{x^2}.$$

Integrating both sides we get:

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C$$
$$\Rightarrow y(x) = Ce^{-x^2} + \frac{1}{2}.$$

Plugging in the initial condition $y(0) = -2$ and solving for C we get:

$$y(0) = C + \frac{1}{2} = -2$$
$$\Rightarrow C = -\frac{5}{2}.$$

So, the solution is:

$$y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}.$$

4. **Higher-Order Linear ODEs and Undetermined Coefficients** (40 points)

For the ordinary differential equation:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x};$$

(a) (15 points) What is the homogeneous solution y_h to this differential equation?

Solution - The corresponding homogeneous equation is:

$$y^{(3)} - 4y'' + 3y' = 0.$$

The corresponding characteristic polynomial is

$$r^3 - 4r^2 + 3r = (r-3)(r-1)r.$$

The roots of the characteristic polynomial are $0, 1, 3$. So, the corresponding homogeneous solution is:

$$y_h(x) = c_1 + c_2 e^x + c_3 e^{3x}.$$

(b) (15 points) Use the method of undetermined coefficients to find a particular solution to the differential equation:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x}$$

from the previous page.

Solution - Our first "guess" based upon the method of undetermined coefficients would be:

$$y_p(x) = A + Be^{2x}.$$

However, the constant term A is not independent of the homogeneous solution. So, we need to multiply it by x to get:

$$y_p = Ax + Be^{2x}.$$

Taking derivatives we get:

$$y_p' = A + 2Be^{2x},$$

$$y_p'' = 4Be^{2x},$$

$$y_p^{(3)} = 8Be^{2x}.$$

Plugging these into the ODE we get:

$$-2Be^{2x} + 3A = 5 + e^{2x}.$$

From this we get $A = 5/3, B = -\frac{1}{2}$, and our particular solution is:

$$y_p(x) = \frac{5}{3}x - \frac{1}{2}e^{2x}.$$

(c) (10 points) Find the solution to the initial value problem:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x};$$

with

$$y^{(2)}(0) = 0, y'(0) = 2, y(0) = 1.$$

Solution - The general solution to the ODE is:

$$y = y_h + y_p = c_1 + c_2e^x + c_3e^{3x} + \frac{5}{3}x - \frac{1}{2}e^{2x}.$$

Taking derivatives we get:

$$y' = c_2e^x + 3c_3e^{3x} + \frac{5}{3} - e^{2x},$$

$$y'' = c_2e^x + 9c_3e^{3x} - 2e^{2x}.$$

Plugging in the initial conditions we get:

$$y(0) = c_1 + c_2 + c_3 - \frac{1}{2} = 1$$

$$y'(0) = c_2 + 3c_3 + \frac{5}{3} - 1 = 2$$

$$y''(0) = c_2 + 9c_3 - 2 = 0.$$

Solving these for the unknown constants we get

$$c_1 = \frac{7}{18}, c_2 = 1, c_3 = \frac{1}{9}.$$

So, the solution to the initial value problem is:

$$y(x) = \frac{7}{18} + e^x + \frac{1}{9}e^{3x} + \frac{5}{3}x - \frac{1}{2}e^{2x}.$$

5. **First-Order Systems of ODEs** (30 points)

Find the general solution to the system of first-order differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

Solution - The eigenvalues of the matrix are:

$$\begin{vmatrix} 1-\lambda & -4 \\ 4 & 9-\lambda \end{vmatrix} = (1-\lambda)(9-\lambda) + 16 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2.$$

So, $\lambda = 5$ is the only eigenvalue. To get a second solution, we'll need to find a generalized eigenvector. So, we'll need a length 2 chain:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1,$$

$$(A - \lambda I)\mathbf{v}_1 = \mathbf{0}.$$

So, $(A - \lambda I)^2 \mathbf{v}_1 = \mathbf{0}$. Calculating $(A - \lambda I)^2$ we get:

$$(A - \lambda I)^2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, *any* vector $\mathbf{v}_2$ that is not already an eigenvector of A will work. Let's make it easy on ourselves and pick

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From this we get

$$\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_2 = \begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

So, our solutions will be:

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{5t},$$
$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{5t}.$$

So, our general solution is:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} -4 \\ 4 \end{pmatrix} e^{5t} + c_2 \left[\begin{pmatrix} -4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{5t}.$$

6. **Solving ODEs with Laplace Transforms** (30 points)

Find the solution to the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$
$$x(0) = x'(0) = 0.$$

The Laplace transform of the left side is:

$$\mathcal{L}(x'' + 4x) = s^2 X(s) - sx(0) - x'(0) + 4X(s) = (s^2 + 4)X(s).$$

The Laplace transform of the right side is:

$$\mathcal{L}(\delta(t) + \delta(t - \pi)) = 1 + e^{-\pi s}.$$

Combining these we get:

$$(s^2 + 4)X(s) = 1 + e^{-\pi s}$$
$$\Rightarrow X(s) = \frac{1 + e^{-\pi s}}{s^2 + 4} = \frac{1}{2}\left(\frac{2}{s^2 + 4}\right) + \frac{1}{2}\left(\frac{2e^{-\pi s}}{s^2 + 4}\right).$$

The inverse Laplace transform is:

$$x(t) = \frac{1}{2}\sin(2t) + u(t - \pi)\frac{1}{2}\sin(2(t - \pi))$$
$$= \frac{1}{2}\sin(2t)(1 + u(t - \pi)).$$

For the last step we note $\sin(2t - 2\pi) = \sin(2t)$.

7. **Convolutions** (15 points)

Calculate the convolution

$$f(t) * g(t)$$

for the functions $f(t) = t + 1$, $g(t) = e^t$.

Solution - The convolution of the functions is:

$$\int_0^t (\tau + 1)e^{t-\tau} d\tau = e^t \left(\int_0^t \tau e^{-\tau} d\tau + \int_0^t e^{-\tau} d\tau \right).$$

The integrals inside the parentheses are:

$$\int_0^t \tau e^{-\tau} d\tau = (-\tau e^{-\tau} - e^{-\tau})|_0^t = -te^{-t} - e^{-t} + 1,$$

$$\int_0^t e^{-\tau} d\tau = -e^{-\tau}|_0^t = -e^{-t} + 1.$$

Plugging these in we get:

$$e^t(-te^{-t} - e^{-t} + 1 - e^{-t} + 1) = 2e^t - t - 2.$$

8. **Fourier Series** (30 points)

The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

Extra Credit (5 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for π.

Solution - We first note that $f(t)$ is odd, so all the a_n terms in the Fourier series will be zero. The period here is $4 = 2L$, so the b_n Fourier coefficients are:

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t) \sin\frac{n\pi t}{2} dt$$

(noting $f(t)$ is odd, so $f(t)\sin\frac{n\pi t}{2}$ is even)

$$= \int_0^2 f(t)\sin\frac{n\pi t}{2} dt$$

$$= \int_0^2 \sin\frac{n\pi t}{2} dt = -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^2 = -\frac{2}{n\pi}((-1)^n - 1)$$

$$= \begin{cases} 0 & n \ even \\ \frac{4}{n\pi} & n \ odd \end{cases}$$

So, our Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n \ odd} \frac{\sin\left(\frac{n\pi t}{2}\right)}{n}.$$

If we plug in $t = 1$ we get:

$$f(1) = 1 = \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\right)$$

$$= \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots),$$

and so,

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$$

which is the famous Leibniz formula for π!