Math 2280 - Final Exam

University of Utah

Summer 2013

Name: ____

This is a two-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction. There are 200 possible points on this exam.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

1. **Basic Definitions** (15 points)

   Circle or state the correct answer to the questions about the following differential equation:

$$(x^3 + 2xe^x - \sin(x))y^{(5)} + x^2y' - e^{-3x}y = \sinh(x^3 + 2)$$

   (3 point) The differential equation is: Linear Nonlinear

   (3 points) The order of the differential equation is:

   (3 points) The corresponding homogeneous equation is:

   For the differential equation:

$$(y')^2 = 2y + 1$$

   (3 point) The differential equation is: Linear Nonlinear

   (3 point) The order of the differential equation is:

2. **Phase Diagrams** (20 points)

   For the autonomous differential equation:

$$\frac{dx}{dt} = x^3 - 4x^2 + 3x$$

   Find all critical points, draw the corresponding phase diagram, and indicate whether the critical points are stable, unstable, or semi-stable.

3. **First-Order Linear ODEs** (20 points)

   Find the solution to the initial value problem:

$$y' + 2xy = x$$

$$y(0) = -2.$$

4. **Higher-Order Linear ODEs and Undetermined Coefficients** (40 points)

For the ordinary differential equation:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x};$$

(a) (15 points) What is the homogeneous solution y_h to this differential equation?

(b) (15 points) Use the method of undetermined coefficients to find a particular solution to the differential equation:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x}$$

from the previous page.

(c) (10 points) Find the solution to the initial value problem:

$$y^{(3)} - 4y'' + 3y' = 5 + e^{2x};$$

with

$$y^{(2)}(0) = 0, y'(0) = 2, y(0) = 1.$$

5. **First-Order Systems of ODEs** (30 points)

   Find the general solution to the system of first-order differential equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & 9 \end{pmatrix} \mathbf{x}.$$

More room, if you need it...

6. **Solving ODEs with Laplace Transforms** (30 points)

   Find the solution to the initial value problem:

$$x'' + 4x = \delta(t) + \delta(t - \pi);$$

$$x(0) = x'(0) = 0.$$

More room, if you need it...

7. **Convolutions** (15 points)

   Calculate the convolution

$$f(t) * g(t)$$

   for the functions $f(t) = t + 1$, $g(t) = e^t$.

8. **Fourier Series** (30 points)

   The values of the periodic function $f(t)$ in one full period are given. Find the function's Fourier series.

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \\ 0 & t = \{-2, 0\} \end{cases}$$

   Extra Credit (5 points) - Use this solution and what you know about Fourier series to deduce the famous Leibniz formula for π.

More room, if you need it...