Math 2280 - Exam 3

University of Utah

Summer 2013

Name: Solutions

This is a one-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

1. (20 points) *Multiple Eigenvalues*

   Find a general solution to the system of differential equations described by:

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}.$$

   Solution - The eigenvalues for the coefficient matrix are:

$$\begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda)+4 = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2.$$

   So, $\lambda = 5$ is the only eigenvalue, and it's repeated. An eigenvector for this eigenvalue must satisfy:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   The vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ works, and there is no other linearly independent eigenvector. So, we want to find a chain of generalized eigenvectors. To do this we note:

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   So, *any* vector that is not an eigenvector will work for the top vector in our chain. If we choose $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get:

$$\mathbf{v}_1 = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

So, our general solution will be:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{5t} + c_2 \left(\begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{5t}.$$

2. *Fundamental Matrices and Matrix Exponentials*

For the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\mathbf{x},$$

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

calculate:

(a) (10 points) A fundamental matrix $\Phi(t)$ for the system. (Problem continued on next page.)

Solution - To calculate a fundamental matrix we'll want to find two linearly independent solutions, and to find two linearly independent solutions we'll need to figure out the eigenvalues of the coefficient matrix:

$$\begin{vmatrix} 2-\lambda & -1 \\ -4 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda = \lambda(\lambda - 4).$$

So, the eigenvalues are $\lambda = 0, 4$.

For the eigenvalue $\lambda = 0$ the eigenvector must satisfy

$$\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 4$ the eigenvector must satisfy

$$\begin{pmatrix} -2 & -1 \\ -4 & -2 \end{pmatrix}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ works.

So, two linearly independent solutions will be $\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{4t}$, and a fundamental matrix is:

$$\Phi(t) = \begin{pmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{pmatrix}.$$

(b) (10 points) The matrix exponential e^{At} for the coefficient matrix $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. (Problem continued on next page.)

Solution - Using the fundamental matrix $\Phi(t)$ we calculated in part (a), we find:

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix},$$

and

$$\Phi(0)^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

So, the matrix exponential e^{At} will be:

$$\begin{aligned} e^{At} = \Phi(t)\Phi(0)^{-1} &= \begin{pmatrix} 1 & e^{4t} \\ 2 & -2e^{4t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{4t} & \frac{1}{4} - \frac{1}{4}e^{4t} \\ 1 - e^{4t} & \frac{1}{2} + \frac{1}{2}e^{4t} \end{pmatrix}. \end{aligned}$$

(c) (10 points) The solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \mathbf{x},$$

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

Solution - Using the matrix exponential calculated in part (b) we get:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{4t} & \frac{1}{4} - \frac{1}{4}e^{4t} \\ 1 - e^{4t} & \frac{1}{2} + \frac{1}{2}e^{4t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + \frac{5}{4}e^{4t} \\ \frac{3}{2} - \frac{5}{2}e^{4t} \end{pmatrix}.$$

3. (10 points) *Laplace Transforms*

Using the definition of the Laplace transform, calculate the Laplace transform of the function:

$$f(t) = te^t.$$

Hint - Integration by parts may be useful.

Solution - Using the definition of the Laplace transform (and some integration by parts) we get:

$$\begin{aligned}
\mathcal{L}(te^t) &= \int_0^\infty e^{-st}te^t dt = \int_0^\infty te^{(1-s)t}dt \\
&= \left.\frac{te^{(1-s)t}}{1-s}\right|_0^\infty + \frac{1}{s-1}\int_0^\infty e^{(1-s)t}dt \\
&= -\left.\frac{e^{(1-s)t}}{(s-1)^2}\right|_0^\infty = \frac{1}{(s-1)^2}.
\end{aligned}$$

Here the integral diverges if $s \leq 1$, so the domain is $s > 1$.

4. (20 points) *Laplace Transforms and Initial Value Problems*

Using Laplace transform methods, find the solution to the initial value problem:

$$x'' - 6x' + 8x = 2;$$

$$x(0) = x'(0) = 0.$$

Solution - If we take the Laplace transforms of the two sides we get:

$$(s^2 - 6s + 8)X(s) = \frac{2}{s}$$

$$\Rightarrow X(s) = \frac{2}{s(s-4)(s-2)}.$$

Taking a partial fraction decomposition of this rational function we get:

$$\frac{2}{s(s-4)(s-2)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-2}$$

$$= \frac{A(s-4)(s-2) + Bs(s-2) + Cs(s-4)}{s(s-4)(s-2)}$$

$$= \frac{(A+B+C)s^2 + (-6A - 2B - 4C)s + 8A}{s(s-4)(s-2)}.$$

Solving for the unknown coefficients we get $A = \frac{1}{4}, B = \frac{1}{4}, C = -\frac{1}{2}$. So, our rational function becomes:

$$\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s-4}\right) - \frac{1}{2}\left(\frac{1}{s-2}\right).$$

The inverse Laplace transform of this funciton is:

$$x(t) = \frac{1}{4} + \frac{1}{4}e^{4t} - \frac{1}{2}e^{2t}.$$

This is the solution to our initial value problem.

5. (20 points) *Solving a System of First-Order Equations*

Find a general solution to the system of first-order equations:

$$\begin{array}{rcrcrcr} x_1' & = & x_1 & + & 2x_2 & + & 2x_3 \\ x_2' & = & 2x_1 & + & 7x_2 & + & x_3 \\ x_3' & = & 2x_1 & + & x_2 & + & 7x_3 \end{array}$$

Solution - The eigenvalues of the coefficient matrix are:

$$\left| \begin{array}{ccc} 1-\lambda & 2 & 2 \\ 2 & 7-\lambda & 1 \\ 2 & 1 & 7-\lambda \end{array} \right|$$

$$= (1-\lambda)(7-\lambda)^2 + 4 + 4 - (1-\lambda) - 4(7-\lambda) - 4(7-\lambda)$$

$$= -\lambda(\lambda - 9)(\lambda - 6).$$

So, the eigenvalues for this matrix are $\lambda = 0, 6, 9$.

For the eigenvalue $\lambda = 0$ an eigenvector must satisfy:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 6$ an eigenvector must satisfy:

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ works.

For the eigenvalue $\lambda = 9$ an eigenvector must satisfy:

$$\begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ works.

So, the general solution to this system of ODEs is:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{6t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{9t}.$$