Math 2280 - Exam 3

University of Utah

Summer 2013

Name: ______________________________

This is a one-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

Things You Might Want to Know

Definitions

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Laplace Transforms

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathcal{L}(\sin(kt)) = \frac{k}{s^2+k^2}$$

$$\mathcal{L}(\cos(kt)) = \frac{s}{s^2+k^2}$$

$$\mathcal{L}(\delta(t-a)) = e^{-as}$$

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s).$$

Translation Formula

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Derivative Formula

$$\mathcal{L}(x^{(n)}) = s^n X(s) - s^{n-1}x(0) - s^{n-2}x'(0) - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0).$$

1. (20 points) *Multiple Eigenvalues*

   Find a general solution to the system of differential equations described by:

$$\mathbf{x}' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \mathbf{x}.$$

More room for Problem 1, if you need it.

2. *Fundamental Matrices and Matrix Exponentials*

For the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \mathbf{x},$$

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

calculate:

(a) (10 points) A fundamental matrix $\Phi(t)$ for the system. (Problem continued on next page.)

(b) (10 points) The matrix exponential e^{At} for the coefficient matrix $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. (Problem continued on next page.)

(c) (10 points) The solution to the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \mathbf{x},$$

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

3. (10 points) *Laplace Transforms*

   Using the definition of the Laplace transform, calculate the Laplace transform of the function:

$$f(t) = te^t.$$

   Hint - Integration by parts may be useful.

4. (20 points) *Laplace Transforms and Initial Value Problems*

   Using Laplace transform methods, find the solution to the initial value problem:

$$x'' - 6x' + 8x = 2;$$

$$x(0) = x'(0) = 0.$$

More room for Problem 4, if you need it.

5. (20 points) *Solving a System of First-Order Equations*

Find a general solution to the system of first-order equations:

$$\begin{aligned} x_1' &= x_1 + 2x_2 + 2x_3 \\ x_2' &= 2x_1 + 7x_2 + x_3 \\ x_3' &= 2x_1 + x_2 + 7x_3 \end{aligned}$$

More room for Problem 5, if you need it.