Math 2280 - Exam 2

University of Utah

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This is a one-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (10 points) *Converting to a First-Order System*

   Convert the following differential equation into an equivalent system of first-order equations:

$$x^{(5)} - t^2 x^{(4)} + \sin(t) x^{(3)} + x'' - 3x' + e^t x = e^{\sin t}.$$

   Solution - We define $x = x_1$, and from this we define:

$$x_1' = x_2,$$

$$x_2' = x_3,$$

$$x_3' = x_4,$$

$$x_4' = x_5,$$

$$x_5' = t^2 x_5 - \sin(t) x_4 - x_3 + 3x_2 - e^t x_1 + e^{\sin(t)}.$$

2. (10 points) *Wronskians*

Use the Wronskian to prove the following functions:

$$f(x) = 1 \qquad g(x) = x \qquad h(x) = x^2$$

are linearly independent on the real line $\mathbb{R}$.

Solution -

$$f(x) = 1 \qquad f'(x) = 0 \qquad f''(x) = 0$$

$$g(x) = x \qquad g'(x) = 1 \qquad g''(x) = 0$$

$$h(x) = x^2 \qquad h'(x) = 2x \qquad h''(x) = 2.$$

The corresponding Wronskian is:

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0.$$

So, as $W(x) \neq 0$ the functions are linearly independent.

3. (10 points) *Existence and Uniqueness*

Upon which intervals are we guaranteed there is a unique solution (given appropriate initial conditions specified on that interval) to the following differential equation:

$$x(x-1)y'' + e^x y' - \sin(x) y = \cos(e^{x^2+5}).$$

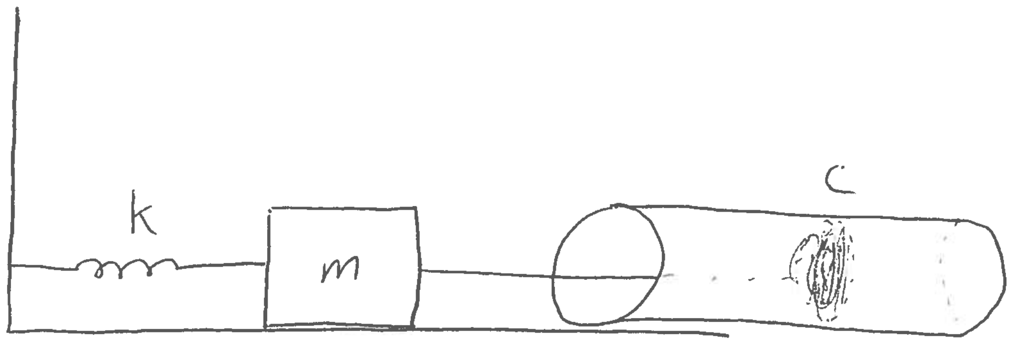
Solution - We can rewrite this differential equation as:

$$y'' + \frac{e^x}{x(x-1)}y' - \frac{\sin(x)}{x(x-1)}y = \frac{\cos(e^{x^2+5})}{x(x-1)}.$$

The coefficient functions are continuous wherever $x(x-1) \neq 0$, which is whenever $x \neq 0, 1$. So, there exists a unique solution on the intervals $(-\infty, 0), (0, 1), (1, \infty)$.

4. (15 points) *Mechanical Systems*

For the mass-spring-dashpot system drawn[1] below:



find the equation that describes its motion with the parameters:

$$m = 3;$$

$$c = 30;$$

$$k = 63;$$

and initial conditions:

$$x_0 = 2 \qquad v_0 = 2.$$

Is the system overdamped, underdamped, or critically damped?

[1]Not very expertly drawn.

Solution - The differential equation that models the motion of this mechanical system is:

$$3x'' + 30x' + 63x = 0.$$

We can rewrite this system as:

$$x'' + 10x' + 21x = 0.$$

The characteristic polynomial for this system is:

$$r^2 + 10r + 21,$$

which has roots $r = \dfrac{-10 \pm \sqrt{10^2 - 4(1)(21)}}{2} = -5 \pm 2 = -7, -3$. As there are two real roots the system is *overdamped*. The solution to our differential equation will be of the form:

$$x(t) = c_1 e^{-7t} + c_2 e^{-3t}.$$

$$v(t) = x'(t) = -7c_1 e^{-7t} - 3c_2 e^{-3t}.$$

If we plug in $x(0) = 2$ and $v(0) = 2$ we get:

$$2 = c_1 + c_2,$$

$$2 = -7c_1 - 3c_2.$$

Solving this system we get $c_1 = -2, c_2 = 4$. So, the motion of the system will be described by the equation:

$$x(t) = -2e^{-7t} + 4e^{-3t}.$$

5. (20 points) *Inhomogeneous Linear Differential Equations*

Find a particular solution to the differential equation:

$$y^{(3)} + y'' = x + e^{-x}.$$

Hint - Find the homogeneous solution first!

Solution - The corresponding homogeneous equation is:

$$y^{(3)} + y'' = 0.$$

This differential equation has characteristic polynomial:

$$r^3 + r^2 = r^2(r + 1).$$

The roots of this polynomial are $r = 0, 0, -1$, where 0 is listed twice as it is a repeated root of multiplicity 2. So, the solution to this differential equation will be:

$$y(x) = c_1 + c_2 x + c_3 e^{-x}.$$

The initial "guess" for our particular solution would be:

$$y_p = A + Bx + Ce^{-x}.$$

However, this won't do at all as no term is linearly independent of our homogeneous solution. To make them so we must multiply the first two terms by x^2, and the final term by x to get:

$$y_p = Ax^2 + Bx^3 + Cxe^{-x}.$$

From this we get:

$$y_p'' = 2A + 6xB + Cxe^{-x} - 2Ce^{-x},$$

$$y_p^{(3)} = 6B - Cxe^{-x} + 3Ce^{-x}.$$

Plugging this into our differential equation we get:

$$y_p^{(3)} + y_p'' = (2A + 6B) + 6Bx + Ce^{-x} = x + e^{-x}.$$

From this we get $C = 1, B = \dfrac{1}{6}, A = -\dfrac{1}{2}$, and our particular solution is:

$$y_p = -\frac{1}{2}x^2 + \frac{1}{6}x^3 + xe^{-x}.$$

6. (20 points) *Endpoint Values*

The eigenvalues for the differential equation below are all nonnegative. First, determine whether $\lambda = 0$ is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

$$y'' + \lambda y = 0;$$

$$y'(0) = 0 \qquad\qquad y(1) = 0.$$

Solution - We first check if $\lambda = 0$ is an eigenvalue. If $\lambda = 0$ the solution to the ODE is:

$$y(x) = Ax + B.$$

If we plug in the endpoint conditions we get:

$$0 = y'(0) = A,$$
$$0 = y(1) = A + B.$$

Solving this system we get $A = 0$ and $B = 0$ is the only solution, so for $\lambda = 0$ there is only the trivial solution, and therefore $\lambda = 0$ is not an eigenvalue.

For $\lambda > 0$ the solution to our differential equation will be (defining $\alpha = \sqrt{\lambda}$, where $\alpha > 0$):

$$y(x) = A\cos(\alpha x) + B\sin(\alpha x),$$
$$y'(x) = -\alpha A\sin(\alpha x) + \alpha B\cos(\alpha x).$$

If we plug in our endpoint conditions we get:

$$0 = y'(0) = \alpha B,$$
$$0 = y(1) = A\cos(\alpha) + B\sin(\alpha).$$

From the first equation we get $B = 0$, and so in order for there to be a non-trivial solution we must have $A \neq 0$, which requires $\cos(\alpha) = 0$. This is true for

$$\alpha = \frac{n\pi}{2} \qquad n \text{ odd.}$$

The corresponding eigenvalues will be:

$$\lambda_n = \frac{n^2\pi^2}{4} \qquad n \text{ odd,}$$

with eigenfunctions

$$y_n = \cos\left(\frac{n\pi}{2}x\right) \qquad n \text{ odd.}$$

7. (15 points) *Euler's Method*

For the differential equation:

$$\frac{dy}{dx} = y^2 - 2y + 3x^2 + 2$$

with $y(0) = 2$ user Euler's method with step size $h = 1$ to estimate $y(2)$.

Solution - For the first step we have:

Step one -

$$f(x_0, y_0) = f(0, 2) = 2^2 - 2(2) + 3(0^2) + 2 = 2,$$

$$x_1 = 1,$$

$$y_1 = y_0 + h * f(x_0, y_0) = 2 + 1 * 2 = 4.$$

Step two -

$$f(x_1, y_1) = f(1, 4) = 4^2 - 2(4) + 3(1^2) + 2 = 13,$$

$$x_2 = 2,$$

$$y_2 = 4 + 1 * 13 = 17 \approx y(2).$$