# Math 2280 - Exam 1

University of Utah

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This is a one-hour exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (30 Points) *Differential Equation Basics*

   **(a)** (5 points) What is the order of the differential equation given below?[1]

   $$y'' \sin(x^2) + (y'')^2 e^{x^3} + 23xy^{(3)}y^2 = 5x^6 + 7x^3 - \arctan x$$

   *Solution* - The highest derivative of $y$ in the differential equation is the third derivative, so the order of the ODE is $3$.

   **(b)** (5 points) Is the differential equation given below linear?

   $$y'' + x^2 y' + e^x y = \cos(\sin(x^2 + 3x + 2))$$

   *Solution* - Yes! The differential equation is linear, as $y$ and all its derivatives appear linearly.

[1] Extra credit - Solve this differential equation! Just kidding. Do not attempt to solve it.

**(c)** (10 points) On what intervals are we guaranteed a unique solution exists for the differential equation below?

$$y' + \frac{y}{x} = \frac{x+3}{x^2-1}$$

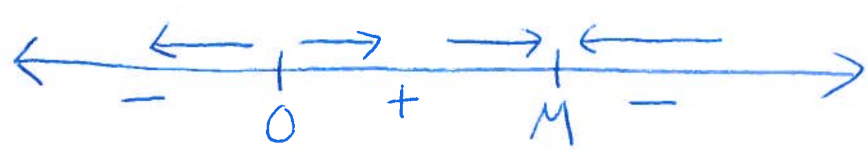
*Solution* - The function $P(x) = \frac{1}{x}$ is continuous for $x \neq 0$, and the function $Q(x) = \frac{x+3}{x^2-1}$ is continuous for $x \neq \pm 1$. So, we're guaranteed a unique solution exists on the four intervals $(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

**(d)** (10 points) Find the critical points for the autonomous equation:

$$\frac{dP}{dt} = kP(M - P).$$

Draw the corresponding phase diagram, and indicate if the critical points are stable, unstable, or semistable.

*Solution* - The critical points are the values of $P$ for which $\frac{dP}{dt} = 0$. These are the roots of $kP(M-P)$, which are $P = 0$ and $P = M$. The phase diagram looks like:




From this phase diagram we can see that $P = 0$ is unstable, while $P = M$ is stable.

2. (25 points) *Separable Equations*

Find the solution to the initial value problem given below.

$$\frac{dy}{dx} = 3x^2(y^2+1) \qquad\qquad y(0) = 1.$$

*Hint* - The integral $\int \frac{du}{1+u^2} = \arctan u + C$ might be useful to you.

*Solution* - We can rewrite the differential equation above as:

$$\frac{dy}{y^2+1} = 3x^2 dx.$$

Taking the antiderivative of both sides, and using the integral from the hint (see, I told you it might be useful) we get:

$$\int \frac{dy}{y^2+1} = \int 3x^2 dx,$$

$$\Rightarrow \arctan y = x^3 + C.$$

Solving for $y$ we get:

$$y(x) = \tan(x^3 + C).$$

The initial condition is $y(0) = 1$, and so:

$$1 = y(0) = \tan C.$$

Now, $\tan C = 1$ when $C = \frac{\pi}{4}$. So, the solution to our initial value problem is:

$$y(x) = \tan\left(x^3 + \frac{\pi}{4}\right).$$

3. (20 points) *Exact Equations*

Find the general solution to the differential equation given below.[2]

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.$$

*Solution* - This is an equation of the form:

$$M(x, y)dx + N(x, y)dy = 0.$$

We want to check if it's an exact equation. To do so, we examine:

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} = \frac{\partial N}{\partial x}.$$

So, it's exact. The solution, $F(x, y)$, will be:

$$F(x, y) = \int (1 + ye^{xy})dx = x + e^{xy} + g(y).$$

To solve for $g(y)$ we calculate:

$$\frac{\partial F}{\partial y} = xe^{xy} + g'(y) = N(x, y) = 2y + xe^{xy}.$$

So, $g'(y) = 2y$, and therefore $g(y) = y^2$.

This means our final solution is:

$$x + y^2 + e^{xy} = C.$$

[2] The title of this problem is a hint.

4. (25 points) *First-Order Linear Equations*

Find a solution to the initial value problem given below, and give the interval upon which you know the solution is unique.

$$xy' = 2y + x^3 \cos x \qquad\qquad y(\pi) = 3\pi^2.$$

*Solution* - We can rewrite the above differential equation as:

$$y' - \frac{2}{x}y = x^2 \cos x.$$

This is a linear first-order differential equation, and the integrating factor will be:

$$\rho(x) = e^{-2\int \frac{dx}{x}} = e^{-2\ln x} = (e^{\ln x})^{-2} = x^{-2} = \frac{1}{x^2}.$$

Multiplying both sides of the above equation by $\frac{1}{x^2}$ we get:

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \cos x.$$

We can rewrite this as:

$$\frac{d}{dx}\left(\frac{1}{x^2}y\right) = \cos x.$$

Taking the antiderivative of both sides we get:

$$\frac{1}{x^2}y = \int \cos x \, dx = \sin x + C.$$

If we solve this for $y$ we get:

$$y(x) = Cx^2 + x^2 \sin x.$$

Plugging in the given initial condition $y(\pi) = 3\pi^2$ gives us:

$$3\pi^2 = y(\pi) = C\pi^2 + \pi^2 \sin(\pi) = C\pi^2.$$

So, $C = 3$, and the solution to our initial value problem is:

$$y(x) = 3x^2 + x^2 \sin x.$$

The function $P(x) = -\frac{2}{x}$ is continuous for $x \neq 0$, while the function $Q(x) = x^2 \cos x$ is continuous for all $x$. So, we know our solution is unique on the interval $x > 0$, which we can also write as $(0, \infty)$.